Chapter 1

Origin

Perturbation theory comprises mathematical methods for finding an approximate solution to a problem, by starting from the exact solution of a related, simpler problem. A critical feature of the technique is a middle step that breaks the problem into "solvable" and "perturbation" parts. Perturbation theory is applicable if the problem at hand cannot be solved exactly, but can be formulated by adding a "small" term to the mathematical description of the exactly solvable problem.

Perturbation theory leads to an expression for the desired solution in terms of a formal power series in some "small" parameter - known as a **perturbation series** - that quantifies the deviation from the exactly solvable problem. The leading term in this power series is the solution of the exactly solvable problem, while further terms describe the deviation in the solution, due to the deviation from the initial problem. Formally, we have for the approximation to the full solution A, a series in the small parameter.

If A_0 would be the known solution to the exactly solvable problem and $A_1, A_2, \ldots$ represents the **higher-order terms** which may be found iteratively by some systematic procedure, for small ϵ these higher-order terms in the series becomes successively smaller. Furthermore, if the small parameter is zero, then the given equation is exactly solvable, and the problem is reduced to finding the asymptotic behavior of the best approximation to the true solution.

An approximate "perturbation solution" is obtained by truncating the series, usually by keeping only the first two terms, the initial solution and the "first order" perturbation correction. If truncated after first term then it is termed as "second order" perturbation correction.[1]

1.1 Historical Development

Initially, Perturbation theory was proposed for the solution of problems in celestial mechanics, particularly to study the motions of planets in the solar system, As the mass of each planet is very less as compared to that of the Sun and the planets are very remote from each other, the gravitational forces between the planets can be neglected, and the planetary motion is taken into consideration to the first approximation as taken along Kepler's orbits (Appendix - A 1.7.1), which are defined by the equations of the two-body problem (Appendix - A 1.7.2), the two bodies being the planet and the Sun. The solution to the two-body problem can be obtained using Kepler's law which suggests that the point mass planet moves on an elliptical path with the Sun at one of the two foci.

The dynamics become extremely complicated when gravitational influence from any other body is added. Since the available astronomical data is of high accuracy, considering how the motion of a planet around the Sun is affected by other planets becomes necessary. This was the origin of the three-body problem (Appendix - B 1.8.1); thus, in studying the system Moon-Earth-Sun the mass ratio between the Moon and the Earth was chosen as the small parameter. Later Poincare[2] showed that the three-body problem does not admit a sufficient number of prime integrals which allow to integration problem. This gave rise to the concept of restricted three body problem.

J. L. Lagrange (25 January 1736 - 10 April 1813), and P. Laplace (23 March 1749 - 05 March 1827) were the first ones to progress over the view that the constants describing the motion of the planet around the sun are perturbed, as they were due to the motion of the other planets. The constants vary as a function of time, hence the name Perturbation theory. Perturbation theory focuses on finding an approximate solution of nearly- integrable systems, i. e., systems which consists of an integrable part and a small perturbation. The core aspect of this theory is to construct a canonical transformation which eliminates the perturbation of higher orders. A typical example of a nearly-integrable system is provide by two-body problem greatly stimulated the development of perturbation theories. The dynamics of the solar system has always been a testing ground for such theories, whose applications extend from the computation of the ephemerides of the natural bodies to developing the trajectories for artificial satellites.

The dynamics of asteroids are driven by the Sun and perturbation by Jupiter, as the Jupiter-Sun mass ratio is approximately 10^{-3}. The solutions to this type of problem motivated the work of scientists around the $XVIII$ and XIX centuries. Indeed, Laverrier (11 March 1811 - 23 September 1877), Delaunary (12 March 1890 - 17 July 1980), Tisserand (13 January 1845 - 20 October 1896) and Poincare (29 April 1854 - 17 July 1912) added to further develop perturbation theories which are at the basis of the study of the dynamics of celestial bodies.

1.2 Discovery of Neptune Planet

Neptune was the first planet to be discovered by using mathematics. Figure 1.1 is the pictorial representation of the solar system. After the discovery of Uranus in 1781, astronomers noticed that the planet was being pulled slightly out of its normal orbit. John Couch Adams of Britain and Urbain Jean Joseph Leverrir of France used mathematics to predict that the gravity from another planet beyond Uranus was affecting the orbit of Uranus. They figured out not only where the planet was, but also how much mass it had. Following the suggestion provided by the theoretical investigation a young astronomer, Johann Gottfried Galle decided to search for the predicted planet and observed Neptune for the first time on 23 September 1846. This discovery represented the first triumph of perturbation theory.

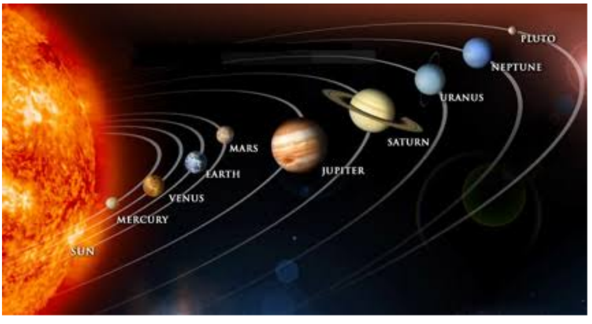

Figure 1.1: Solar System

Similar to the discovery of Neptune, Pluto was discovered in 1930 by

Clyde Tombaugh. Astronomers noticed that the orbits of Neptune and Uranus were being affected by the gravity of an unknown object in the solar system. Clyde Tombaugh carefully studied images of the night sky, and after a lot of hard work, he finally discovered Pluto. Interestingly Clyde Tombaugh was only 24 years old when he made this discovery.

Well-developed perturbation methods were adopted and adapted to solve new problems arising during the development of quantum mechanics in $20^(th)$-century atomic and subatomic physics. Paul Dirac developed perturbation theory in 1927 to evaluate when a particle would be emitted in radioactive elements. It was later named Fermi's golden rule.

1.3 Mixed term theory

The mathematical challenge involved initially in the development of the theory was that the term in the expansion consisted of the time parameter t independent of the sine and cosine functions. The contribution of such terms to the series is significant only when t is very large(of the order of several hundred years), even then only the first approximation is obtained instead of the accurate planetary motion. The secular terms i.e., the terms of the form At^n appears because the frequency of the motion(rotation) of the planet under study depends on the respective frequencies of other planets. If this kind of relation is allowed then both secular and mixed terms $Bt\cos(\omega t+\psi)$ appear in the solution. Thus, in the framework of perturbation theory the relation

$$\omega = \omega_0 + \epsilon\,\omega_1 \tag{1.1}$$

permits the following expansion with respect to $\epsilon(\epsilon \ll 1)$:

$$\sin\omega t = \sin\omega_0 t + \epsilon\,\omega_1\,t\cos\omega_0 t + \dots \tag{1.2}$$

The mixed term in this equation is obtained by expanding oscillations of frequency (1.1) by oscillations with frequency ω_0 .

Lindstedt (27 June 1854 - 16 May 1939), Paul Guldin (1577 - 1643), Ch. Delaunay (1890-1980),B. Bohlin, and S. Newcomb's (March 12, 1835 - July 11, 1909) work led to the development of special methods in Perturbation theory, which eliminates the secular terms and hence, it permits one to obtain a purely trigonometric solution. The expansion of the frequencies affected by the secular terms is no more an expansion with respect to the small parameter which means such expansion does not contain zero approximation frequencies instead it contains the frequencies which to some extent have been redefined or in modern physical terms, it is renormalized. Therefore, in perturbation theory, every member of the power series consisting of powers of a small parameter is convergent.

1.4 Small denominator problem

The method for the construction of a special kind of periodic solution was proposed by H. Poincare and A.M. Lyapunov and this method is efficient not only in the problems related to celestial mechanics but also used in the theory of differential equations in general.

The order of perturbation can be reduced using the method of successive canonical change of variables. This method also allows to take advantage of better convergence(superconvergence) in order to overcome the divergence of the series which appears due to the presence of small denominators in each series no matter what order, with the help of suitable canonical transformation.

1.5 Theory of Oscillations

Subsequent advances in perturbation theory are connected with the development of the theory of oscillations, especially with the development of the theory of non-linear oscillations. Non-linear ordinary differential equation of Rayleigh (1.3)

$$\ddot{x} + F(\dot{x}) + x = 0, \qquad \dot{x} = \frac{dx}{dt} \tag{1.3}$$

where $F(u)$ satisfies the assumption $uF(u) < 0$ for small (u) and $uF(u) > 0$ for large (u) desirable auto-oscillation with one degree of freedom. The special case of Rayleigh equation is when

$$F(u) = -\mu(u - \frac{u^2}{3}), \qquad \mu = \text{constant}$$

It is known as The van der Pol equation, a special case of equation (1.3) in order to solve the equation

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0, \qquad \mu = const > 0, \qquad \dot{x}(t) \equiv \frac{dx}{dt} \tag{1.4}$$

Van der Pol proposed the method of slowly varying coefficients that resembled the method used by Lagrange in solving problems of celestial mechanics. This method of slowly varying coefficients comprises of representation of the equation in which functions harmonically oscillate and their amplitudes and phases slowly vary as the function of the parameter t.

N. M. Krylov and N. N. Bogolyubov developed general theory of non-linear oscillations and used asymptotic methods in perturbation theory. Consider equation

$$\frac{d^2x}{dt^2} + \omega^2 x = \epsilon f \left(x, \frac{dx}{dt}, \epsilon \right) \tag{1.5}$$

In the case $\epsilon = 0$, equation 1.5 describes the oscillations that are purely harmonic which have a constant amplitude and a uniformly recurring phase. If $\epsilon \neq 0$, i.e. in the presence of a non-linear perturbation solution of equation 1.5 involves overtones, the dependence of instantaneous frequency on the amplitude, and a systematic increase/ decrease of the amplitude depending on the input or output of energies due to perturbing forces.

$$x = a \cos\psi + \epsilon u_1(a, \psi) + \epsilon^2 u_2(a, \psi) + \ldots \tag{1.6}$$

where $u_i(a, \psi), i = 1, 2, \ldots$ represent periodic functions of angle ψ with period 2π, The values of a and ψ are taken as functions of time and are defined by the differential equations.

$$\left. \begin{aligned} \tfrac{da}{dt} &= \dot{a} = \epsilon A_1(a) + \epsilon^2 A_2(a) + \ldots \\ \tfrac{d\psi}{dt} &= \dot{\psi} = \omega + \epsilon B_1(a) + \epsilon^2 B_2(a) + \ldots \end{aligned} \right\} \tag{1.7}$$

Thus, the problem is reduced to the choice of suitable expressions for the functions $u_i(a, \psi), A_i(a), B_i(a), i = 1, 2, \ldots$ so that the expression (1.6),along with a and ψ after replacing by functions dependent on time as in equation 1.7, becomes the solution of the original equation 1.5. Additional conditions are imposed to ensure that the secular terms (terms of the form At^n) do not appear in the solution.
If in the formal series 1.6 the expansion is cut-off after $(m+1)^{th}$ terms, one obtains the m-th approximation; this approximation is asymptotic i.e. if m is fixed and $epsilon \rightarrow 0$, then the expression 1.6 approximates the exact solution of 1.5. Van der Pol equation and the first approximation equations are identical.

1.6 Averaging Method

Differential equations describing oscillatory processes and containing a "small" parameter may often be reduced to standard form

$$\frac{dX_s}{dt} = \epsilon X_s(t, x_1, \dots), x_n), \ s = 1, \dots, n \tag{1.8}$$

where ϵ is a small positive parameter. Several problems in physics and technology can be reduced to this form. Averaging method, a special method of approximation is developed for systems of differential equations as in 1.8. This method suggests that if values of ϵ are sufficiently small on a finite interval, the change of variables can be used to obtain the average equations.

$$x_i = \xi_i + \epsilon X_i$$

can be used to obtain the averaged equations

$$\frac{d\xi_s}{dt} = \epsilon X_s, 0(\xi_1, \dots, \xi_n), \ s = 1, \dots, n \tag{1.9}$$

where

$$X_s, 0(\xi_1, \dots, \xi_n) = \lim_{T \to \infty} \frac{1}{T} \int_0^T X_s(t, \xi_1, \dots, \xi_n) dt$$

The number of criteria for the existence and stability of auto-oscillatory systems can also be obtained by the averaging method.
Equation 1.9 give the estimates of the difference $|X_i - \epsilon_i|$ over a time interval of length L/ϵ.

1.7 Appendix - A

1.7.1 The approximate nature of Kepler's laws and two body Problem

The constraints placed on the force for Kepler's laws to be derivable from Newton's laws were that the force must be directed toward a central fixed point and that the force must decrease as the inverse square of the distance. In actuality, however, the sun. which serves as the source of the major force, is not fixed but experiences small accelerations because of the planets, in accordance with Newton's second and third laws. Furthermore, the planets attract one another, so that the total force on a planet is not just that due to the sun, other planets perturb the elliptical motion that would have occurred for a particular planet if that planet had been the only orbiting an isolated Sun. Keplar's laws, therefore, are only approximate. The motion of the Sun itself means that, even when the attractions by other planets are neglected, Kepler's third law must be replaced by $(M + m_i)\,\tau^2 \propto a^3$, where m_i is one of the planetary masses and M is the Sun's mass. The Kepler's law are such good approximations to the actual planetary results from the fact that all the planetary masses are very small compared to that of the Sun. The perturbations of the elliptic motion are therefore small, and the coefficient $M + m_i \equiv M$ for all the planetary masses m_i means that Kepler's third law is very close to being true.

Newton's second law for a particular mass is a second-order differential equation that must be solved for whatever forces may act on the body if its position as a function of time is to be deduced. The exact solution of this equation, which resulted in a derived trajectory that was an ellipse, parabola, or hyperbola, depended on the assumption that there were only two point particles by the inverse square force. Hence, this "gravitational two-body problem" has exact Solution that reproduces Kepler's laws. If one or more additional bodies also interact with the original pair through their mutual gravitational interactions, no exact solution for differential equations of motion of any of the bodies involved can be obtained. As was noted above, however, the motion of a planet is almost elliptical, since all masses involved are small compared to the Sun. it is then convenient to treat the motion of a particular planet as slightly perturbed elliptical motion and to determine the changes in the parameters of the ellipse that result from the small forces as time progresses. It is the elaborate developments of various perturbation theories and their applications to approximate the exact motion of celestial bodies that has occupied celestial mechanicians since Newton's time.

1.7.2 Two-body problem

A problem dealing with the motion of two material points P_1 and P_2 with masses m_1 and m_2 respectively, moving in three-dimensional Euclidean space E^3 when acted upon by the mutual Newton attracting forces. The problem is special case of the n-body problem, which may be described by a system of ordinary differential equations of order $6n$, and has 10 independent integrals: 6 of motion of the centre of inertia, 3 of law of areas (equivalently, conservation of angular momentum) and 1 of energy conservation. The two-body problem also has three Laplace integrals (one of which is independent of the preceding ones) and is completely integrable.

The integration of the two-body problem is more conveniently effected in a special system of coordinates, in which these integrals are employed. If the origin of the Cartesian coordinates x, y, z is placed at the centre of masses $(m_1r_1 + m_2r_2)/(m_1 + m_2)$ and the axis z is directed along the relative angular momentum vector, then the motion of the relative position vector $r_1 - r_2 = (x, y, z)$ takes place in the plane $z = 0$ and satisfies the system

$$\mu\ddot{x} = -fxr^{-3}, \qquad \mu\ddot{y} = -fyr^{-3}, \qquad (1)$$

where $r = \sqrt{x^2 + y^2}$, $\mu = m_1m_2/(m_1 + m_2)$ is the reduced mass and f is the gravitational constant. The system (1) has four integrals:

$x\dot{y} - y\dot{x} = x$ (law of areas),

$\frac{1}{2}\mu(\dot{x}^2 + \dot{y}^2) - fr^{-1} = h$ (energy)

$\mu^2c\dot{y} - \mu fxr^{-1} = \lambda_1$ and $\mu^2c\dot{x} + \mu fyr^{-1} = \lambda_2$ (Laplace),

which are interconnected by the relation

$\lambda_1^2 + \lambda_2^2 = 2\mu^3hc^2 + \mu^2f^2$

here

$$c^2 = \lambda_1x + \lambda_2y + \mu r, \qquad (2)$$

i.e. the orbits of the relative position vector are conical sections with parameter $p = c^2/\mu$, major semi-axis $a = -\mu/(2h)$, eccentricity $e = \mu^{-1}\sqrt{1 + 2hc^2}$, longitude of pericentre $\omega(\lambda_1 = \mu e\cos\omega, \lambda_2 = \mu e\sin\omega)$, and with the focus at the coordinate origin. the location of the relative positive vector on the orbit is determined by the true anomaly v, counted from the direction towards the pericentre; (2) then implies that $r = p/(1 + e\cos v)$. if $c \neq 0$, three types of orbits are possible:

1. If $h < 0$, they are ellipses.
2. if $h > 0$, they are hyperbolas.
3. If $h = 0$, they are parabolas.

If $c = 0$, the motion is rectilinear. The two-body problem describes an unperturbed Kepler motion of a planet with respect to the Sun or of a satellite respect to a planet, etc.

1.8 Appendix B

1.8.1 Three-body problem

The problem on the motion of three bodies, regarded as material points, mutually attracting one another according to Newton's law of gravitation (cf. Newton laws of mechanics). The classical example of the three-body problem is that of the motion of the Sun-Earth-Moon system. The three-body problem consists in finding the general solution of the system of differential equations

$$m_i \frac{d^2 x_i}{\partial t^2} = \frac{\partial U}{\partial x_i}, \qquad m_i \frac{d^2 y_i}{\partial t^2} = \frac{\partial U}{\partial y_i}, \qquad m_i \frac{d^2 z_i}{dt^2} = \frac{\partial U}{\partial z_i}, \qquad i = 1, 2, 3,$$

where x_i, y_i, z_i are the rectangular coordinates of the body M_i in some absolute coordinate frame with fixed axes, t is the time, m_i is the mass of M_i, and U is the potential, which depends only on the mutual distances between the points. The function U is defined by the relation

$$U = f\left(\frac{m_1 m_2}{\Delta_{12}} + \frac{m_2 m_3}{\Delta_{23}} + \frac{m_3 m_2}{\Delta_{13}}\right), f > 0,$$

where the mutual distances $\Delta_{ij}, i, j = 1, 2, 3$ are given by the formula

$$\Delta_{ij} = \Delta_{ji} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2}$$

From the properties of the potential one can derive ten first integrals of the equations of motion in the absolute system of coordinates, Six of them, called the integrals of motion of the centre of mass, determine the uniform rectangular motion of the centre of mass of the three bodies. The three integrals of the angular momentum fix the value and the direction of the angular momentum of the three body system. The energy integral defines the constant magnitude of total energy of the system. H. Bruns (1887) proved that the equations of motion of the three-body problem have no other first integrals expressible in terms of algebraic functions of the coordinate and their derivatives. H. Poincare (1889) further proved that the equations of motion of the three-body problem do not have transcendental integrals expressible in terms of single-values analytic functions. C. Sundman (1912) found the general solution of the problem in the form of power series in a certain regularizing variable, converging at each instant. However, the Sundman series proved to be completely useless for qualitative investigations as well as for practical computations due to its extremely slow convergence.

The equation of three-body problem admits five particular solutions, in which all three material points are in some fixed plane. Here, the configuration of the three bodies remain fixed and they describe Kepler trajectories with a common focus at the centre of mass of the system. Two of the particular solutions corresponds to the case when the three bodies form an equilateral triangle at all times. This is the so-called triangular solution of the three-body problem, or the Lagrange solutions corresponding to three bodies on one straight line are called the rectilinear solutions, or the Euler solutions.

For the general solution of the three-body problem, final motions have been studied in detail, that is, the limiting properties of the motion as $t \to +\infty$ and $t \to -\infty$.

A particular case of the three-body problem is the so-called restricted three-body problem, which is obtained from the general three-body problem in case the mass of one of the three bodies is so small that its influence on the motion of the other two bodies can be neglected. In this case, the bodies M_1 and M_2 with finite masses m_1 and m_2 move under the action of their mutual attraction along Kepler orbits. In the right-handed rectangular coordinate system $G\,\xi\,\eta\,\zeta$ with origin G at the centre of mass of M_1 and M_2, with axis ξ directed along the line joining M_1 and M_2 and axis ζ perpendicular to the plane of their motion, the motion of the third body M_3 of small mass is described by the following differential equation:

$$\ddot{\xi} - 2\dot{\nu} \quad \dot{\eta} - \dot{\nu}^2 \quad \xi - \ddot{\nu} \quad \eta = \frac{\partial W}{\partial \xi},$$

$$\ddot{\eta} - 2\dot{\nu} \quad \dot{\xi} - \dot{\nu}^2 \quad \eta - \ddot{\nu} \quad \xi = \frac{\partial W}{\partial \eta},$$

$$\ddot{\zeta} = \frac{\partial W}{\partial \zeta},$$

where $W = f\left(\frac{m_1}{r_1} + \frac{m_2}{r_2}\right),$

ν is the true anomaly of the Kepler motion of M_1 and M_2, and r_1 and r_2 are the distances of M_3 from M_1 and M_2, respectively. In the case of the circular restricted three-body problem,
$\dot{\nu} = n = const, \qquad \ddot{\nu} = 0,$

the equation of motion of M_3 have also a first integral, called the jacobi integral, of the form

$$\dot{\xi}^2 + \dot{\eta}^2 + \dot{\xi}^2 = n^2(\xi^2 + \eta^2) + 2f\left(\frac{m_1}{r_1} + \frac{m_2}{r_2}\right) + C$$

where C is an arbitrary constant. The surface defined by the equation

$$n^2(\xi^2 + \eta^2)2f\left(\frac{m_1}{r_1} + \frac{m_2}{r_2}\right) + C = 0$$

is called the surface of zero velocity and is remarkable in that it determines the regions of possible motions of M_3 relative to M_1 and M_2. The restricted three-body problem has particular solutions similar to those of the general three-body problem. The position of the body with a small mass in these particular solutions is called the points of libration.

For the restricted three-body problem, various classes of periodic motion have been investigated.