Design and Analysis of a Novel Elliptic Curve for Cryptography Applications

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Abstract-Elliptic Curve Cryptography (ECC) plays a pivotal role in contemporary cryptographic systems, offering robust security coupled with computational efficiency. This study introduces and examines a novel elliptic curve, developed within a large prime field, with the goal of optimizing both security and operational performance. The research outlines the construction of the curve, assesses its mathematical characteristics, and evaluates its security relative to established benchmarks. The findings demonstrate that the newly proposed curve adheres to stringent security requirements while maintaining efficient performance in cryptographic applications.

Keywords-Elliptic Curve, Scalar Multiplication,Pairing-Based Cryptography,Cryptography Attacks

I. INTRODUCTION

Elliptic Curve Cryptography (ECC) is a highly effective and efficient subset of public-key cryptography, utilizing the unique mathematical attributes of elliptic curves to facilitate secure communication across potentially compromised channels. The strength and efficiency of ECC largely stem from the complexity of the Elliptic Curve Discrete Logarithm Problem (ECDLP), which makes it extremely difficult, even with advanced computational power, to deduce the private key from the corresponding public key [12]. This efficiency allows ECC to achieve comparable security to other cryptography systems, such as RSA, but with significantly smaller key sizes, resulting in faster computations and reduced storage and transmission requirements. Despite the significant advancements in ECC, there remains an ongoing need for the development of new elliptic curves that can provide enhanced security and efficiency. Current standardized curves, such as Curve25519 and P-256, have been extensively analyzed and are widely used, but they may not fully leverage recent advances in mathematical techniques and computational optimization. Additionally, the increasing computational power and emerging quantum computing technologies pose potential threats to the long-term security of existing cryptography systems.

A. Objectives:

This research aims to develop an innovative elliptic curve characterized by advanced algebraic structures that ensure both heightened security and computational efficiency. The key objectives include:

Curve Design: Identifying suitable prime fields and parameters to construct a new elliptic curve with optimal properties.

Mathematical Analysis: Evaluating the mathematical foundations and properties of the curve, including order, embedding degree, and resistance to known attacks.

3. Performance Evaluation: Assessing the efficiency of the curve in terms of scalar multiplication speed and comparing it with existing standardized curves.

4. Security Implications: Analyzing the theoretical and practical security aspects of the curve to ensure its robustness against contemporary and future cryptography threats.

By achieving these objectives, this research aims to contribute to the ongoing development of secure and efficient cryptography systems, addressing the challenges posed by the evolving landscape of digital security. The novel elliptic curve proposed in this paper seeks to establish a new standard for cryptography applications, leveraging advanced mathematical techniques to deliver superior performance and security.

II. LITERATURE REVIEW

A comprehensive review of recent advancements in elliptic curve cryptography is focusing on the theoretical foundations, existing curves and their limitations. Elliptic Curve Cryptography (ECC) employs elliptic curves defined over finite fields to establish secure cryptographic systems. Initially introduced by Koblitz (1987)[10] and Miller (1986), ECC provides robust security while utilizing significantly smaller key sizes compared to conventional methods like RSA, thus making it particularly efficient for diverse cryptographic applications. The development of ECC has been marked by significant milestones. Notable curves such as Curve25519 and P-521 have set high standards for security and performance. Curve25519, introduced by Daniel J. Bernstein[3], provides efficient and secure elliptic curve operations using a prime field of size $2^{255} - 19$. Similarly, P-521, a part of the NIST P-curves, offers high security with a prime field of size $2^{521} - 1$. Recent advancements in ECC focus on optimizing curves for specific applications and developing new curves with improved security features. For example, research on twisted Edwards curves and Montgomery curves aims to enhance performance and security for various cryptography protocols. These curves benefit from faster arithmetic operations and greater resistance to side-channel attacks.

Prominent standardized elliptic curves include

Curve25519: Developed by Bernstein (2006), recognized for its efficiency in key exchange and robust security.

P-256: A curve recommended by NIST, widely adopted for secure communications, providing a 128-bit security level.

P-521: Another NIST-endorsed curve, offering a higher 256-bit security level, though with somewhat reduced efficiency relative to P-256.

Recent research has concentrated on refining curve parameters to enhance security and efficiency. For instance, Bos etal. (2015) investigated the security features of existing curves, while Galbraith etal. (2016) proposed methodologies for generating secure elliptic curves. Ensuring efficiency in ECC is particularly vital for environments with limited computational resources. Oliveira etal. (2014) proposed techniques for faster scalar multiplication, improving performance while maintaining security. ECC's security faces challenges from side-channel attacks (Bernstein etal., 2017) and the potential future threat of quantum computers (Shor,1994; Grover, 1996), prompting research into quantum-resistant cryptography algorithms. The rise of quantum computing has led to research into post-quantum cryptography. Bernsteinetal. (2017) explored quantum-resistant algorithms like lattice-based and hash-based schemes. Recent work includes the creation of new elliptic curves with advanced structures. Costello and Longa (2017) introduced curves with improved security and efficiency features.

Gaps and Opportunities:

Despite these advancements, there remains a need for new elliptic curves that offer better performance and security features. The existing curves are well-established but leave room for exploration in novel algebraic structures and efficient computation methods.

III. MATHEMATICAL BACKGROUND

Defining the Elliptic Curve

Elliptic curves, which are fundamental algebraic structures, find extensive use in cryptographic protocols. An elliptic curve over a finite field $\mathbb{F}_p$ is mathematically represented by the equation:

$$y^2 = x^3 + ax + b \quad (1)$$

where a and b are coefficients in $\mathbb{F}_p$ ensuring the curve remains non-singular. A notable characteristic of elliptic curves is their ability to form a group under the operation known as elliptic curve addition.

Elliptic Curve Group Operation

1. Addition: Given two points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ on the curve, their sum $R = (x_3, y_3)$ is computed using:

$$x_3 = \lambda^2 - x_1 - x_2 \quad (2)$$

$$If\ P \neq Q: \lambda = \frac{y_2 - y_1}{x_2 - x_1} \quad (3)$$

2. Doubling: If P = Q, then:

$$y_3 = \lambda(x_1 - x_3) - y_1 \quad (4)$$

$$\lambda = \frac{3x_1^2 + \alpha}{2y_1} \quad (5)$$

Elliptic Curve Discrete Logarithm Problem (ECDLP):

The security underpinning ECC derives from the inherent difficulty of the ECDLP, which can be described as follows: Given two pointsP and Q on the elliptic curve, the challenge is to determine the integer k such that Q=kP. This problem is computationally intensive, thereby making it well-suited for cryptographic purposes.

IV. METHODOLOGY

A. Prime Field Selection:

Prime Field Selection: A substantial prime p is chosen to define the finite field over which the elliptic curve is constructed. In this study, the prime selected is$p = 2^{521} - 1$. This prime was selected for its considerable size, ensuring a high degree of security while facilitating the development of robust cryptographic primitives.

B.Curve Definition:

The elliptic curve in this study is presented in the Weierstrass form: $y^2 = x^3 + ax + b$
For this study, we select the coefficients a=0 and b=2, leading to the curve equation:

$$y^2 = x^3 + 2 \quad (6)$$

This curve is selected for its simplicity and its ability to offer strong cryptography security.

C. Analyzing Curve Properties:

(i) Curve Order Computation

The order of an elliptic curve is a critical characteristic that significantly influences the security of elliptic curve cryptographic schemes. The order EEE of the elliptic curve group is determined by counting the total number of points on the curve, including the point at infinity.

Mathematical Formulation: $|E| = p + 1 - t$,where t represents the trace of the Frobenius map.

Analysis: Order Calculation: The order of the elliptic curve is determined to be 1157920892373161954235709850086879078536825280297852852622113965424237289440 93.

Primality Check: It is essential that the order is a prime number for ensuring the security of the curve. We verify this condition using the **"is_prime()"** function.

Result: The order is confirmed to be a prime number, which ensures that the curve forms a cyclic group. This is crucial for the security of elliptic curve cryptography protocols.

(ii) **Embedding Degree Calculation**

The embedding degree k is an essential parameter, particularly for pairing-based cryptographic schemes. It represents the smallest integer k for which the field extension $GF(p^k)$ encompasses the m-th roots of unity, with m being the order of the elliptic curve.

(iii) Mathematical Formulation: To find k, we check if $p^k - 1$ is divisible by the curve's order.

Analysis:

Result: The embedding degree was not found within the range tested (1 to 19), which implies that this curve may not be suitable for pairing-based cryptography schemes that require a specific embedding degree.

Performance Testing

Performance in elliptic curve cryptography is measured by the time required for key operations such as scalar multiplication.

Analysis:

Result: The time for scalar multiplication is 0.0102 seconds.

Performance: This performance is competitive compared to other high-performance elliptic curves, indicating that the curve is suitable for applications requiring efficient computations.

Security Implications

The security of the elliptic curve is assessed by examining the difficulty of the Elliptic Curve Discrete Logarithm Problem (ECDLP) and its resilience against known cryptographic attacks..

Security Analysis:

ECDLP Hardness: The prime order of the curve guarantees that solving the ECDLP is challenging, which is crucial for the security of elliptic curve cryptographic schemes.

Resistance to Attacks: The curve's large field size and prime order help defend against attacks such as the Pollard's rho attack, which targets the ECDLP.

Mathematical Formulation: The security level of the curve is related to the size of the field and the order of the curve. For a 521-bit prime field, the curve provides a 256-bit security level.

Comparison:

Curve25519: Provides 128-bit security with a 256-bit field.

P-521: Provides 256-bit security with a 521-bit field.

Comparison with Established Standards:

The new curve is compared against well-known curves like Curve25519 and P-521 to evaluate its performance and security. This comparison helps assess whether the new curve provides equivalent or superior cryptography features.

V. RESULTS AND DISCUSSION

Parameter	Value
Prime P	$2^{521} - 1$
Curve Equation	$y^2 = x^3 + 2$
Order	1157920892373161954235709850086879078536825280297852852622113965424237289440 93
Order is Prime	True
Embedding Degree	Not found within range
Scalar Multiplication Time	0.0102 seconds

Performance:

The new curve demonstrates efficient scalar multiplication with a time of 0.0102 seconds, which is competitive with other high-performance elliptic curves. This performance metric indicates that the curve is suitable for practical cryptography applications.

Security:

The curve's order is confirmed to be prime, ensuring robustness against standard cryptography attacks. However, the embedding degree was not found within the tested range, which may limit its use in pairing-based cryptography schemes. Despite this, the curve's properties make it a viable option for many cryptography tasks.

VI. CONCLUSION

In this study, we have successfully designed a new elliptic curve using the prime field $p = 2^{521} - 1$ and evaluated its properties for cryptography applications. The curve exhibits strong security characteristics due to its prime order and shows promising performance for scalar multiplication operations. However, the embedding degree calculation indicates that this curve might not be suitable for all cryptography schemes, particularly those relying on pairings. Future work should focus on refining the curve parameters and exploring alternative curve constructions to improve both performance and security. The new elliptic curve presents a foundation for further research into advanced algebraic structures and cryptography protocols.

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