Some fractional integral inequalities for strongly (α, m)-convex functions via unified Mittag-Leffler functions

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Abstract— **In this paper, we study several kinds of fractional integral operators containing generalized and extended Mittag-Leffler functions in their kernels. We establish fractional bounds for strongly (α;m)-convex functions via unified Mittag-Leffler functions. We also prove Hermite-Hadamard type inequality by using strongly (α;m)-convex functions close to symmetric functions.**

***Keywords*— Hermite-Hadamard inequality, Fractional integral operators, Strongly (α;m)-convex functions, Symmetry**

I. Introduction

The notion of convexity is very important and basic in the theory of integral inequalities. It has numerous extensions and generalizations that have been extremely beneficial to many branches of mathematics. Strongly convex function was introduced and studied by Karamardian [13]. Lara *et al.* [17] gave strongly m-convex function and proved that if two functions are m-convex then their maximum is also m-convex. For more details on generalizations and extensions of convex functions, we refer, [18, 19, 26, 27, 29, 32].

The theory of fractional calculus has gained huge attention during the last few years due mainly to its demonstrated applications in the field of pure and applied science, see, [3, 5, 30]. Kilbas *et al.* [15] discussed different type of fractional integrals and presented the fundamental existence and uniqueness theorem for ordinary fractional differential equations.. Further, Deng and Wang [6] proved Hermite-Hadamard inequalities for twice differentiable (α, m)-logarithmically convex function using Riemann-Liouville fractional integral. For more fractional integral inequalities, see, [12, 25, 28, 29].

In 1903, Mittag-Leffler [20] defined and studied a function named as Mittag-Leffler function, which is direct generalization of the exponential series. In fractional calculus, this function serves the same purpose as the exponential function does in conventional calculus. We may refer to readers [2, 8, 23, 24]. Kilbas and Saigo [16] studied composition properties of integral operators involving generalized Mittag-Leffler function in the kernels. Further Andric *et al.* [2] introduced an extended form of generalized Mittag-Leffler function and established Opial-type inequalities for this function.

Recently, Donge *et al.* [7] gave strongly (α, m)-convex function and established a new refinement of the bounds of generalized fractional integral operators. Gao *et al.* [11] proved the boundedness of a generalized integral operator by using unified Mittag-Leffler function and also constructed various type of Minkowski and Reverse Minkowski type inequality. Further, Nonlaopon *et al.* [22] defined the bounds of integral operator involving unified Mittag-Leffler function with the use of (α, m)-convexity.

Inspired by the above works, we establish fractional bounds via unified Mittag-Leffler function using strongly (α, m)-convex function. Furthermore, we get the H-H type inequality with the help of the condition close to symmetry about the interval's midpoint. The results established in this paper give various kinds of well-known fractional and conformable integral inequalities.

The work in this paper as follows: In Section 2, we give some basic definitions and results which are necessary for our main results. In Section 3, we obtain fractional bounds for strongly (α, m)-convex functions via unified Mittag-Leffler functions. We also obtain the Hermite-Hadamard type inequality with the help of the conditions close to symmetry. In Section 4, we give the conclusions and future directions of our work.

II. Preliminaries

In this section, we give some necessary definitions for our main results.

Definition 1. [14] *A function* $\Upsilon: [0, d] \to \mathfrak{R}$, $d > 0$ *is called strongly* (α, m)*-convex, where* $(\alpha, m) \in [0,1]^2$*, if* $\Upsilon(\xi x + m(1-\xi)y) \leq \xi^\alpha \Upsilon(x) + m(1-\xi^\alpha)\Upsilon(y) - \sigma m \xi^\alpha (1-\xi^\alpha)(y-x)^2$ *(2.1) satisfy* $\forall$ *x, y* $\in [0, d]$ *and* $\xi \in [0,1]$.

Remark 1. (i) *If we take* $\alpha = 1$*, then (*2.1*), we get strongly m-convex function.*

(ii) *If we take* $\sigma = 0$ *then (*2.1*), we get* (α, m)*-convex function.*

(iii) *If we take* $(\alpha, m) = (1, m)$*, then (*2.1*), we get m-convex function.*

(iv) *If we take* (α, m)=(1,1)*, then (*2.1*), we get convex function.*

(v) *If we take* (α, m)=(1,0)*, then (*2.1*), we get star-shaped function.*

(vi) *If we take* σ=0 *and* (α, m)=(1,0)*, then (*2.1*), we get convex function.*

Definition 2. [15] *Let* $\Upsilon \in L_1[c, d]$*. The left-sided and right-sided Riemann-Liouville (R-L) fractional integrals are defined as follows:*

$$I_{c^+}^{\mu} \Upsilon(x) = \frac{1}{\Gamma(\mu)} \int_c^x (x-\xi)^{\mu-1} \Upsilon(\xi) d\xi, \ x > c \tag{2.2}$$

And

$$I_{d^-}^{\mu} \Upsilon(x) = \frac{1}{\Gamma(\mu)} \int_x^d (\xi - x)^{\mu-1} \Upsilon(\xi) d\xi, \ x < d, \tag{2.3}$$

Where $\Gamma(\mu) = \int_0^\infty e^{-z} z^{\mu-1} dz$.

Definition 3. [21] *Suppose* $\Upsilon \in L_1[c, d]$*. Then, K-fractional R-L integrals of order* μ*, where* $\Re(\mu) > 0, K > 0$*, are*

$${}_kI_{c^+}^{\mu} \Upsilon(x) = \frac{1}{K\Gamma_k(\mu)} \int_c^x (x-\xi)^{\frac{\mu}{k}-1} \Upsilon(\xi) d\xi, \ x > c \tag{2.4}$$

And

$${}_kI_{d^-}^{\mu} \Upsilon(x) = \frac{1}{K\Gamma_k(\mu)} \int_x^d (\xi - x)^{\frac{\mu}{k}-1} \Upsilon(\xi) d\xi, \ x < d, \tag{2.5}$$

where $\Gamma_k(.)$ *is defined as follows:* $\Gamma_k(\mu) = \int_0^\infty t^{\mu-1} e^{-\frac{t^k}{k}} dt, \ \Re(\mu) > 0.$

Generalization of R-L fractional integral using a monotonically increasing function is stated as:

Definition 4. [15] *Suppose an integrable function* $\Upsilon: [c, d] \to \Re$, *and* Λ *be a positive increasing function on (c, d], have a continuous derivative* Λ' *on* (c, d). *For a function* Υ *the left-sided and right-sided fractional integrals with respect to another function* Λ *on* $[c, d]$ *of order* μ *where* $\Re(\mu) > 0$ *are defined as:*

$$I_{c^+}^{\mu,\Lambda}\Upsilon(x) = \frac{1}{\Gamma(\mu)}\int_c^x \Lambda'(\xi)(\Lambda(x) - \Lambda(\xi))^{\mu-1}\Upsilon(\xi)d\xi,\ x > c \tag{2.6}$$

And

$$I_{d^-}^{\mu,\Lambda}\Upsilon(x) = \frac{1}{\Gamma(\mu)}\int_x^d \Lambda'(\xi)(\Lambda(\xi) - \Lambda(x))^{\mu-1}\Upsilon(\xi)d\xi,\ x < d. \tag{2.7}$$

Definition 5. [1] *In the above Definition 4 the K-analogue is defined as:*

$${}_kI_{c^+}^{\mu,\Lambda}\Upsilon(x) = \frac{1}{K\Gamma_k(\mu)}\int_c^x \Lambda'(\xi)(\Lambda(x) - \Lambda(\xi))^{\frac{\mu}{k}-1}\Upsilon(\xi)d\xi,\ x > c \tag{2.8}$$

And

$${}_kI_{d^-}^{\mu,\Lambda}\Upsilon(x) = \frac{1}{K\Gamma_k(\mu)}\int_x^d \Lambda'(\xi)(\Lambda(\xi) - \Lambda(x))^{\frac{\mu}{k}-1}\Upsilon(\xi)d\xi,\ x < d. \tag{2.9}$$

In 1903, Mittag-Leffler [20] defined a function named as Mittag-Leffler function as follows:

$$E_\alpha(z) = \sum_{l=0}^{\infty}\frac{z^l}{\Gamma(\alpha l + 1)}\ \alpha \in \mathbb{C}, \Re(\alpha) > 0, z \in \mathbb{C}.$$

Definition 6. [2] *Suppose* $\omega, \alpha, \beta, \gamma, \theta, \eta \in \mathbb{C}$, $\Re(\alpha), \Re(\beta), \Re(\gamma) > 0, \Re(\theta) > \Re(\eta) > 0$ *with* $\rho \geq 0, \delta > 0$ *and* $0 < k \leq r + \Re(\alpha)$. *Let* $\Upsilon \in L_1[c, d]$ *and* $x \in [c, d]$. *Then, the generalized fractional integral operators* $\epsilon_{c^+,\alpha,\beta,\gamma}^{\omega,\eta,\theta,k,r}\Upsilon$ *and* $\epsilon_{d^-,\alpha,\beta,\gamma}^{\omega,\gamma,\delta,k,c}\Upsilon$ *are defined by*

$$\left(\epsilon_{c^+,\alpha,\beta,\gamma}^{\omega,\eta,\theta,k,r}\Upsilon\right)(x;s) = \int_c^x (x-\xi)^{\alpha-1}\,E_{\alpha,\beta,\gamma}^{\eta,\theta,k,r}(\omega(x-\xi)^\mu; s)\Upsilon(\xi)d\xi \tag{2.10}$$

And

$$\left(\epsilon_{d^-,\alpha,\beta,\gamma}^{\omega,\eta,\theta,k,r}\Upsilon\right)(x;s) = \int_x^d (\xi-x)^{\alpha-1}\,E_{\alpha,\beta,\gamma}^{\eta,\theta,k,r}(\omega(\xi-x)^\mu; s)\Upsilon(\xi)d\xi, \tag{2.11}$$

where $E_{\alpha,\beta,\gamma,c}^{\eta,\theta,k,r}(z;s) = \sum_{l=0}^{\infty}\frac{\beta_s(\eta+lk,\theta-\eta)(\theta)_{lk}z^l}{\beta(\eta,\theta-\eta)(\gamma)_{lr}\Gamma(\alpha l+\beta)}$

called extended generalized Mittag-Leffler function.

Definition 7. [9] *Suppose* $\Upsilon, \Lambda: [c, d] \to \Re$, $0 < c < d$ *such as* Υ *is positive and* $\Upsilon \in L_1[c, d]$ *and* Λ *are differentiable and strictly increasing. Let* $\frac{P}{x}$ *be an increasing function on* $[c, \infty)$ *and* $\omega, \alpha, \gamma, \eta, \theta \in \mathbb{C}$, $\Re(\alpha), \Re(\gamma) > 0, \Re(\theta) > \Re(\eta) > 0, \alpha, \delta > 0, s \geq 0$ *and* $0 < k \leq \delta + \alpha$. *Then, for* $x \in [c, d]$, *the left and right integral operator stated as:*

$$({}_\Lambda^PF_{c^+,\alpha,\beta,\gamma}^{\omega,\eta,\rho,\theta,k,\delta}\Upsilon)(x;s) = \int_c^x R_x^\xi\left(E_{\alpha,\beta,\gamma}^{\eta,\theta,k,\delta}\Lambda; P\right)\Upsilon(\xi)d(\Lambda(\xi)) \tag{2.12}$$

and $({}_\Lambda^PF_{d^-,\alpha,\beta,\gamma}^{\omega,\eta,\rho,\theta,k,\delta}\Upsilon)(x;s) = \int_x^d R_\xi^x\left(E_{\alpha,\beta,\gamma}^{\eta,\theta,k,\delta}\Lambda; P\right)\Upsilon(\xi)d(\Lambda(\xi)),$ (2.13)

where $R_x^\xi\left(E_{\alpha,\beta,\gamma}^{\eta,\theta,k,\delta}\Lambda; P\right) = \frac{P(\Lambda(x)-\Lambda(\xi))}{\Lambda(x)-\Lambda(\xi)}E_{\alpha,\beta,\gamma}^{\eta,\theta,k,\delta}(\omega(\Lambda(x)-\Lambda(\xi))^\mu; s).$

Definition 8. [33] *For* $\underline{a} = (a_1, a_2, \ldots, a_n)$, $\underline{b} = (b_1, b_2, \ldots, b_n)$, $\underline{e} = (e_1, e_2, \ldots, e_n)$, *where* $a_i, b_i, e_i \in \mathbb{C}$; $i = 1,2,3, \ldots, n$ *such that* $\Re(a_i), \Re(b_i), \Re(e_i) > 0$ *for all i. Let* $\alpha, \beta, \gamma, \delta, \mu, x, \eta, \rho, \theta, t \in \mathbb{C}, min\big(\Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\delta), \Re(\theta)\big) > 0$ *and* $K \in (0,1) \cup \mathbb{N}$ *with* $s \geq 0$, *Let* $K + \Re(\rho) < \Re(\delta + x + \alpha)$ *with* $Im(\rho) = Im(\delta + x + \alpha)$; *then*

$$M^{\eta,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\delta,\mu,x}\left(z; \underline{a}, \underline{b}, \underline{e}, s\right) = \sum_{l=0}^{\infty} \frac{\prod_{i=1}^{r} \beta_s(b_i,a_i)(\eta)_{\rho l}(\theta)_{kl} z^l}{\prod_{i=1}^{r} \beta(e_i,a_i)(\gamma)_{\delta l}(\mu)_{\nu l}\Gamma(\alpha l+\beta)}, \tag{2.14}$$

where β_s *is stated as:*

$$\beta_s(x, y) = \int_0^1 \xi^{x-1}\,(1-\xi)^{y-1} e^{-\left(\frac{s}{\xi(1-\xi)}\right)} d\xi. \tag{2.15}$$

Definition 9. [33] *Let* $\Upsilon \in L_1[c, d]$. *Then* $\forall\ x \in [c, d]$, *The fractional integral operator involving the unified Mittag-Leffler function* $T^{\eta,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\delta,\mu,x}\left(z; \underline{a}, \underline{b}, \underline{e}, s\right)$ *is defined by*

$$\left(A^{\omega,\eta,\rho,k,r}_{c^+,\alpha,\beta,\gamma,\delta,\mu,x}\Upsilon\right)\left(x; \underline{a}, \underline{b}, \underline{e}, s\right) = \int_c^x (x-\xi)^{\alpha-1}\, T^{\eta,\rho,k,r}_{\alpha,\beta,\gamma,\delta,\mu,x}\left(\omega(x-\xi)^{\mu}; \underline{a}, \underline{b}, \underline{e}, s\right)\Upsilon(\xi)d\xi \tag{2.16}$$

$$\left(A^{\omega,\eta,\rho,k,r}_{d^-,\alpha,\beta,\gamma,\delta,\mu,x}\Upsilon\right)\left(x; \underline{a}, \underline{b}, \underline{e}, s\right) = \int_x^d (\xi-x)^{\alpha-1}\, T^{\eta,\rho,k,r}_{\alpha,\beta,\gamma,\delta,\mu,x}\left(\omega(x-\xi)^{\mu}; \underline{a}, \underline{b}, \underline{e}, s\right)\Upsilon(\xi)d\xi, \tag{2.17}$$

$a_i = l, s = 0$ *and* $\Re(s) > 0$ *in (*2.8*), we obtain the fractional integral operator related with the generalized Q function as follows:*

$$({}_Q A^{\omega,\eta,\rho,k,r}_{c^+,\alpha,\beta,\gamma,\delta,\mu,x}\Upsilon)\left(x; \underline{a}, \underline{b}\right) = \int_c^x (x-\xi)^{\alpha-1}\, Q^{\eta,\rho,k,r}_{\alpha,\beta,\gamma,\delta,\mu,x}\left(\omega(x-\xi)^{\mu}; \underline{a}, \underline{b}\right)\Upsilon(\xi)d\xi, \tag{2.18}$$

$$({}_Q A^{\omega,\eta,\rho,k,r}_{d^-,\alpha,\beta,\gamma,\delta,\mu,x}\Upsilon)\left(x; \underline{a}, \underline{b}\right) = \int_x^d (\xi-x)^{\alpha-1}\, Q^{\eta,\rho,k,r}_{\alpha,\beta,\gamma,\delta,\mu,x}\left(\omega(\xi-x)^{\mu}; \underline{a}, \underline{b}\right)\Upsilon(\xi)d\xi, \tag{2.19}$$

where $Q^{\eta,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\delta,\mu,x}\left(z; \underline{a}, \underline{b}\right) = \sum_{l=0}^{\infty} \frac{\prod_{i=1}^{r} \beta(b_i,l)(\eta)_{\rho l}(\theta)_{kl} z^l}{\prod_{i=1}^{r} \beta(a_i,l)(\gamma)_{\delta l}(\mu)_{\nu l}\Gamma(\alpha l+\beta)}$.

This function is defined by Bhatnagar et al. [4] named as generalized Q *function.*

Definition 10. [11] *Let* $P \in L_1[c, d]$, $0 < c, d < \infty$ *be a positive function and let* $\Lambda: [c, d] \to \Re$ *be a differentiable and stictly increasing function. Suppose* $\frac{\xi}{x}$ *be an increasing function on* $[c, \infty)$ *for all* $x \in [c, d]$, *unified integral operator is:*

$$({}^P_\Lambda A^{\omega,\eta,\rho,\theta,k,r}_{c^+,\alpha,\beta,\gamma,\delta,\mu,x}\Upsilon)(x; s) = \int_c^x D_x^{\xi}\left(T^{\eta,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,x}\Lambda; P\right)\Upsilon(\xi)d\big(\Lambda(\xi)\big), \tag{2.20}$$

$$({}^P_\Lambda A^{\omega,\eta,\rho,\theta,k,r}_{d^-,\alpha,\beta,\gamma,\delta,\mu,x}\Upsilon)(x; s) = \int_x^d D_{\xi}^{x}\left(T^{\eta,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,x}\Lambda; P\right)\Upsilon(\xi)d\big(\Lambda(\xi)\big), \tag{2.21}$$

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$$where\ D_x^{\xi}\left(T^{\eta,\rho,\theta,k,\delta}_{\alpha,\beta,\gamma,\mu,x}\Lambda;P\right)=\frac{P(\Lambda(x)-\Lambda(\xi))}{\Lambda(x)-\Lambda(\xi)}T^{\eta,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\delta,\mu,x}\left(\omega(\Lambda(x)-\Lambda(\xi))^{\mu},\underline{a},\underline{b},\underline{e};s\right). \tag{2.22}$$

Remark 2. (i) *If* $r=1$, $b_1=\eta+lk$, $a_1=\theta-\eta, e_1=\eta, \rho=x=0$ *in (*2.20*) and (*2.21*), we get the unified integral operator which is given in Definition* 7*;*
(ii) *If* $P(x)=x^{\alpha}$ *and* $\Lambda(x)=x$, *we obtained the operators which are given in Definition* 9.
(iii) *If* $P(x)=x^{\alpha}$ *for* $\alpha>0$, *in (*2.20*), we get the required operator which is introduced by Gao et al. [11].*
(iv) *If r=1,* $b_1=\eta+lk, a_1=\theta-\eta, e_1=\eta, \rho=x=0, P(x)=x^{\alpha}$ *and* $\Lambda(x)=x$ *in (*2.20*) and (*2.21*), we get the required operator which is given in Definition* 6.

Definition 11. Taking $a_i=l, s=0$ *and* $\Re(\rho)>0$ *in (*2.20*) and (*2.21*) we obtain the fractional integral operator related with the generalized* Q *function:*

$$({}^{\Lambda}_{Q}A^{p,\omega,\eta,\rho,\theta,k,r}_{c^+,\alpha,\beta,\gamma,\delta,\mu,x}\Upsilon)\left(x;\underline{a},\underline{b}\right)=\int_c^x D_x^{\xi}\left(Q^{\eta,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,x}\Lambda;P\right)\Upsilon(\xi)d(\Lambda(\xi)),$$

$$({}^{\Lambda}_{Q}A^{P\omega,\eta,\rho,\theta,k,r}_{d^-,\alpha,\beta,\gamma,\delta,\mu,x}\Upsilon)\left(x;\underline{a},\underline{b}\right)=\int_x^d D_{\xi}^{x}\left(Q^{\eta,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,x}\Lambda;P\right)\Upsilon(\xi)d(\Lambda(\xi)),$$

$$where\ D_x^{\xi}\left(Q^{\eta,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,x}\Lambda;P\right)=\frac{P(\Lambda(x)-\Lambda(\xi))}{\Lambda(x)-\Lambda(\xi)}Q^{\eta,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\delta,\mu,x}\left(\omega(\Lambda(x)-\Lambda(\xi))^{\mu},\underline{a},\underline{b}\right).$$

The following Lemma will be necessary for proving our results:

Lemma 2.1. [7] *Let* $\Upsilon\in[c,d]\to\Re$ *be strongly* (α,m)*-convex function,* $0<c<md$. *If* $\Upsilon(x)=\Upsilon\left(\frac{c+md-x}{m}\right)$ *and* $(\alpha,m)\in[0,1]^2$ *with* $m\in(0,1]$, *then,*

$$\Upsilon\left(\frac{c+md}{2}\right)\le\Upsilon(x)\left(\frac{1}{2^{\alpha}}+m\left(1-\frac{1}{2^{\alpha}}\right)\right)-\frac{\sigma}{m}\left(\frac{2^{\alpha}-1}{2^{2\alpha}}\right)(c+md-x-mx)^2. \tag{2.25}$$

III. Main Results

In this section, we prove fractional bounds for strongly (α,m)-convex functions via unified Mittag-Leffler function.

Theorem 3.1. *Let* $\Upsilon\in L_1[c,d]$ *be an integrable strongly* (α,m)*-convex function. Suppose* $\frac{P}{x}$ *be an increasing function on* $[c,d]$ *and let* $\Lambda:[c,d]\to\Re$ *be a differentiable and strictly increasing function. Then, for all* $x\in[c,d]$, *we get the required inequalities involving the unified Mittag-Leffler function* $T^{\eta,p,\theta,k,r}_{\alpha,\beta,\gamma,\delta,\mu,\nu}\left(z;\underline{a},\underline{b},\underline{e},s\right)$ *satisfying all the convergence conditions:*

$$\begin{aligned}
&({}^{P}_{\Lambda}A^{\omega,\eta,\rho,\theta,k,r}_{c^+,\alpha,\beta,\gamma,\mu,x}\Upsilon)(x;s)\\
&\le D_x^{c}\left(T^{\eta,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,x}\Lambda;P\right)\left[\left(m\Upsilon\left(\frac{x}{m}\right)\Lambda(x)-\Upsilon(c)\Lambda(c)\right)\right.\\
&-\frac{\Gamma(\alpha+1)}{(x-c)^{\alpha}}\left(m\Upsilon\left(\frac{x}{m}\right)-\Upsilon(c)\right)I^{\alpha}_{c^+}\Lambda(x)\\
&\left.-\sigma m\left(\frac{x}{m}-c\right)^2\frac{1}{(x-c)^{\alpha}}\left[\Gamma(\alpha+1)I^{\alpha}_{c^+}+\Gamma(2\alpha+1)I^{2\alpha}_{c^+}\right]\Lambda(x)\right],
\end{aligned} \tag{3.1}$$

$$
\begin{aligned}
&({}_{\Lambda}^{P}A_{d^{-},\alpha,\beta,\gamma,\mu,x}^{\omega,\eta,\rho,\theta,k,r}\Upsilon)(x,s)\\
&\leq D_{d}^{x}\left(T_{\alpha,\beta,\gamma,\mu,x}^{\eta,\rho,\theta,k,r}\Lambda;P\right)\left[\left(\Upsilon(d)\Lambda(d)-m\Upsilon\left(\frac{x}{m}\right)\Lambda(x)\right)\right.\\
&-\frac{\Gamma(\alpha+1)}{(d-x)^{\alpha}}\left(\Upsilon(d)-m\Upsilon\left(\frac{x}{m}\right)\right)I_{d^{-}}^{\alpha}\Lambda(x)\\
&-\sigma m\left(\frac{x}{m}-d\right)^{2}\frac{1}{(d-x)^{\alpha}}\left[\Gamma(\alpha+1)I_{d^{-}}^{\alpha}+\Gamma(2\alpha+1)I_{d^{-}}^{2\alpha}\right]\Lambda(x)]. \qquad (3.2)
\end{aligned}
$$

*By adding (*3.1*) and (*3.2*), we get*

$$
\begin{aligned}
&({}_{\Lambda}^{P}A_{c^{+},\alpha,\beta,\gamma,\mu,x}^{\omega,\eta,\rho,\theta,k,r}\Upsilon)(x;s)+({}_{\Lambda}^{P}A_{d^{-},\alpha,\beta,\gamma,\mu,x}^{\omega,\eta,\rho,\theta,k,r}\Upsilon)(x;s)\\
&\leq D_{x}^{c}\left(T_{\alpha,\beta,\gamma,\mu,x}^{\eta,\rho,\theta,k,r}\Lambda;P\right)\left[\left(m\Upsilon\left(\frac{x}{m}\right)\Lambda(x)\right.\right.\\
&\left.-\Upsilon(c)\Lambda(c)\right)-\frac{\Gamma(\alpha+1)}{(x-c)^{\alpha}}\left(m\Upsilon\left(\frac{x}{m}\right)-\Upsilon(c)\right)I_{c^{+}}^{\alpha}\Lambda(x)\\
&-\sigma m\left(\frac{x}{m}-c\right)^{2}\frac{1}{(x-c)^{\alpha}}\left[\Gamma(\alpha+1)I_{c^{+}}^{\alpha}+\Gamma(2\alpha+1)I_{c^{+}}^{2\alpha}\right]\Lambda(x)]\\
&+D_{d}^{x}\left(T_{\alpha,\beta,\gamma,\mu,x}^{\eta,\rho,\theta,k,r}\Lambda;P\right)\left[\left(\Upsilon(d)\Lambda(d)-m\Upsilon\left(\frac{x}{m}\right)\Lambda(x)\right)\right.\\
&-\frac{\Gamma(\alpha+1)}{(d-x)^{\alpha}}\left(\Upsilon(d)-m\Upsilon\left(\frac{x}{m}\right)\right)I_{d^{-}}^{\alpha}\Lambda(x)\\
&-\sigma m\left(\frac{x}{m}-d\right)^{2}\frac{1}{(d-x)^{\alpha}}\left[\Gamma(\alpha+1)I_{d^{-}}^{\alpha}+\Gamma(2\alpha+1)I_{d^{-}}^{2\alpha}\right]\Lambda(x)]. \qquad (3.3)
\end{aligned}
$$

Proof. After considering P and Λ, we have the following inequality:

$$\frac{P\big(\Lambda(x)-\Lambda(\xi)\big)}{\Lambda(x)-\Lambda(\xi)}\leq\frac{P\big(\Lambda(x)-\Lambda(c)\big)}{\Lambda(x)-\Lambda(c)}, \qquad (3.4)$$

for all $\xi\in[c,x]$ and $x\in[c,d]$.
Multiplying $T_{\alpha,\beta,\gamma,\delta,\mu,x}^{\eta,\rho,\theta,k,r}\left(\omega(\Lambda(x)-\Lambda(\xi))^{\mu};\underline{a},\underline{b},\underline{e},s\right)\Lambda'(\xi)$ in (3.4) and from (2.22), we get

$$
\begin{aligned}
&D_{x}^{\xi}\left(T_{\alpha,\beta,\gamma,\mu,x}^{\eta,\rho,\theta,k,r}\Lambda;P\right)\Lambda'(\xi)\\
&\leq\frac{P\big(\Lambda(x)-\Lambda(c)\big)}{\Lambda(x)-\Lambda(c)}T_{\alpha,\beta,\gamma,\delta,\mu,x}^{\eta,\rho,\theta,k,r}\left(\omega(\Lambda(x)-\Lambda(\xi))^{\mu};\underline{a},\underline{b},\underline{e},s\right)\Lambda'(\xi). \qquad (3.5)
\end{aligned}
$$

By using

$$
\begin{aligned}
&T_{\alpha,\beta,\gamma,\delta,\mu,x}^{\eta,\rho,\theta,k,r}\left(\omega(\Lambda(x)-\Lambda(\xi))^{\mu};\underline{a},\underline{b},\underline{e},s\right)\\
&\leq T_{\alpha,\beta,\gamma,\delta,\mu,x}^{\eta,\rho,\theta,k,r}\left(\omega(\Lambda(x)-\Lambda(c))^{\mu};\underline{a},\underline{b},\underline{e},s\right), \qquad (3.6)
\end{aligned}
$$

we get the following inequality:

$$D_{x}^{\xi}\left(T_{\alpha,\beta,\gamma,\mu,x}^{\eta,\rho,\theta,k,r}\Lambda;P\right)\Lambda'(\xi)\leq D_{x}^{c}\left(T_{\alpha,\beta,\gamma,\mu,x}^{\eta,\rho,\theta,k,r}\Lambda;P\right)\Lambda'(\xi). \qquad (3.7)$$

Strongly (α,m)-convex function Υ satisfies the folowing inequality:

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$$\begin{aligned} \Upsilon(\xi) \quad &\leq (\frac{x-\xi}{x-c})^{\alpha}\Upsilon(c) + m\left(1-(\frac{x-\xi}{x-c})^{\alpha}\right)\Upsilon\left(\frac{x}{m}\right) \\ &-\sigma m(\frac{x-\xi}{x-c})^{\alpha}\left(1-(\frac{x-\xi}{x-c})^{\alpha}\right)(\frac{x}{m}-c)^{2}. \end{aligned} \tag{3.8}$$

Multiplying (3.7) and (3.8) then integrating over (c, x), we get

$$\begin{aligned} &\int_{c}^{x} D_{x}^{\xi}\left(T_{\alpha,\beta,\gamma,\mu,x}^{\eta,\rho,\theta,k,r}\Lambda; P\right)\Upsilon(\xi)d\big(\Lambda(\xi)\big) \\ &\leq D_{x}^{c}\left(T_{\alpha,\beta,\gamma,\mu,x}^{\eta,\rho,\theta,k,r}\Lambda; P\right)[\Upsilon(c)\int_{c}^{x}(\frac{x-\xi}{x-c})^{\alpha}d\big(\Lambda(\xi)\big) \\ &+m\Upsilon\left(\frac{x}{m}\right)\int_{c}^{x}\left(1-(\frac{x-\xi}{x-c})^{\alpha}\right)d\big(\Lambda(\xi)\big) \\ &-\sigma m\left(\frac{x}{m}-c)^{2}\int_{c}^{x}\left((\frac{x-\xi}{x-c})^{\alpha}-(\frac{x-\xi}{x-c})^{2\alpha}\right)d\big(\Lambda(\xi)\big)\right]. \end{aligned} \tag{3.9}$$

We use Definition 10 and integrating by parts, we get

$$\begin{aligned} &({}_{\Lambda}^{P}A_{c^{+},\alpha,\beta,\gamma,\mu,x}^{\omega,\eta,\rho,\theta,k,r}\Upsilon)(x;s) \\ &\leq D_{x}^{c}\left(T_{\alpha,\beta,\gamma,\mu,x}^{\eta,\rho,\theta,k,r}\Lambda; P\right)[m\Upsilon\left(\frac{x}{m}\right)\Lambda(x)-\Upsilon(c)\Lambda(c) \\ &-\frac{\Gamma(\alpha+1)}{(x-c)^{\alpha}}\left(m\Upsilon\left(\frac{x}{m}\right)-\Upsilon(c)\right)I_{c^{+}}^{\alpha}\Lambda(x) \\ &-\sigma m(\frac{x}{m}-c)^{2}\left(\frac{\Gamma(\alpha+1)}{(x-c)^{\alpha}}I_{c^{+}}^{\alpha}\Lambda(x)-\frac{\Gamma(2\alpha+1)}{(x-c)^{2\alpha}}I_{c^{+}}^{2\alpha}\Lambda(x)\right)]. \end{aligned} \tag{3.10}$$

Now, for $\xi \in (x, d]$ and $x \in [c, d]$, the following inequality holds:

$$D_{x}^{\xi}\left(T_{\alpha,\beta,\gamma,\mu,x}^{\eta,\rho,\theta,k,r}\Lambda; P\right)\Lambda'(\xi) \leq \Lambda_{d}^{x}\left(T_{\alpha,\beta,\gamma,\mu,x}^{\eta,\rho,\theta,k,r}\Lambda; P\right)\Lambda'(\xi). \tag{3.11}$$

From strongly (α, m)-convex function Υ, we have

$$\begin{aligned} \Upsilon(\xi) \quad &\leq (\frac{\xi-x}{d-x})^{\alpha}\Upsilon(d) + m(1-\left(\frac{\xi-x}{d-x})^{\alpha}\right)\Upsilon\left(\frac{x}{m}\right) \\ &-\sigma m(\frac{\xi-x}{d-x})^{\alpha}(1-\left(\frac{\xi-x}{d-x})^{\alpha}\right)(\frac{x}{m}-d)^{2}. \end{aligned} \tag{3.12}$$

Now, adopting the similar procedure as we did for (3.7) and (3.8), the following inequality we can get from (3.11) and (3.12):

$$
\begin{aligned}
&({}^{P}_{\Lambda}A^{\omega,\eta,\rho,\theta,k,r}_{d^-,\alpha,\beta,\gamma,\mu,x}\Upsilon)(x;s)\\
&\leq D^{x}_{d}\left(T^{\eta,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,x}\Lambda;P\right)[\left(\Upsilon(d)\Lambda(d)-m\Upsilon\left(\frac{x}{m}\right)\Lambda(x)\right)\\
&-\frac{\Gamma(\alpha+1)}{(d-x)^{\alpha}}\left(\Upsilon(d)-m\Upsilon\left(\frac{x}{m}\right)\right)I^{\alpha}_{d^-}\Lambda(x)\\
&-\sigma m\left(\frac{x}{m}-d\right)^{2}\frac{1}{(d-x)^{\alpha}}[\Gamma(\alpha+1)I^{\alpha}_{d^-}+\Gamma(2\alpha+1)I^{2\alpha}_{d^-}]\Lambda(x)].
\end{aligned}
\tag{3.13}
$$

By adding (3.10) and (3.13), (3.3) can be obtained.

Corollary 3.2. *If we consider α=1 in equation (3.3) then we obtained the result for strongly m-convex functions:*

$$
\begin{aligned}
&({}^{P}_{\Lambda}A^{\omega,\eta,\rho,\theta,k,r}_{c^+,\alpha,\beta,\gamma,\mu,x}\Upsilon)(x;s)+({}^{P}_{\Lambda}A^{\omega,\eta,\rho,\theta,k,r}_{d^-,\alpha,\beta,\gamma,\mu,x}\Upsilon)(x;s)\\
&\leq D^{c}_{x}\left(T^{\eta,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,x}\Lambda;\Upsilon\right)[\left(m\Upsilon\left(\frac{x}{m}\right)\Lambda(x)-\Upsilon(c)\Lambda(c)\right)\\
&-\frac{1}{(x-c)}\left(m\Upsilon\left(\frac{x}{m}\right)-\Upsilon(c)\right)\int_{c}^{x}\Lambda(\xi)d\xi-\sigma m(\frac{x}{m}-c)^{2}(\frac{1}{(x-c)}\int_{c}^{x}\Lambda(\xi)d\xi\\
&-\frac{2}{(x-c)^{2}}\int_{c}^{x}(x-\xi)\Lambda(\xi)d\xi)]+D^{x}_{d}\left(T^{\eta,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,x}\Lambda;P\right)\\
&\times[\left(\Upsilon(d)\Lambda(d)-m\Upsilon\left(\frac{x}{m}\right)\Lambda(x)\right)\\
&-\frac{1}{(d-x)}\left(\Upsilon(d)-m\Upsilon\left(\frac{x}{m}\right)\right)\int_{x}^{d}\Lambda(\xi)d(\xi)\\
&-\sigma m\left(\frac{x}{m}-d)^{2}\left(\frac{1}{(d-x)}\int_{x}^{d}\Lambda(\xi)d(\xi)+\frac{2}{(d-x)^{2}}\int_{x}^{d}(\xi-x)\Lambda(\xi)d(\xi)\right)\right].
\end{aligned}
\tag{3.14}
$$

Remark 3. *If $\sigma=0$, then above theorem reduces to Theorem 1 of [22] .*

Remark 4. *If $(\alpha,m)=(1,1)$ and $\sigma=0$, then above theorem reduces to Example 3 of [22].*

Remark 5. *If $\alpha=1$ and $\sigma=0$, then above theorem reduces to Example 2 of [22].*

The following result given below we have obtain the upper and lower bounds of the sum of operators (2.20) and (2.21) in the form of a Hadamard inequality.

Theorem 3.3. *After considering Theorem 3.1, in addition, if*

$\Upsilon(x)=\Upsilon\left(\frac{c+md-x}{m}\right)$, *then*

$$
\begin{aligned}
\frac{2^{\alpha}}{1+m(2^{\alpha}-1)}&[\Upsilon\left(\frac{c+md}{2}\right)\left(({}^{P}_{\Lambda}A^{\omega,\eta,\rho,\theta,k,r}_{d^-,\alpha,\beta,\gamma,\mu,x}1\right)(c;s)+\quad+({}^{P}_{\Lambda}A^{\omega,\eta,\rho,\theta,k,r}_{c^+,\alpha,\beta,\gamma,\mu,x}1)(d;s))\\
&+\frac{\sigma}{m}\left(\frac{2^{\alpha}-1}{2^{2\alpha}}\right)(({}^{P}_{\Lambda}A^{\omega,\eta,\rho,\theta,k,r}_{d^-,\alpha,\beta,\gamma,\mu,x}(c+md-x-mx)^{2})(c;s)\\
&+({}^{P}_{\Lambda}A^{\omega,\eta,\rho,\theta,k,r}_{c^+,\alpha,\beta,\gamma,\mu,x}(c+md-x-mx)^{2})(d;s))]
\end{aligned}
$$

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$$\leq ({}^{P}_{\Lambda}A^{\omega,\eta,\rho,\theta,k,r}_{d^-,\alpha,\beta,\gamma,\mu,x}\Upsilon)(c;s) + ({}^{P}_{\Lambda}A^{\omega,\eta,\rho,\theta,k,r}_{c^+,\alpha,\beta,\gamma,\mu,x}\Upsilon)(d;s)$$

$$\leq 2D^c_d\left(T^{\eta,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,x}\Lambda;P\right)[\left(\Upsilon(d)\Lambda(d) - m\Upsilon\left(\frac{c}{m}\right)\Lambda(c)\right)$$

$$-\frac{\Gamma(\alpha+1)}{(d-c)^\alpha}\left(\Upsilon(d) - m\Upsilon\left(\frac{x}{m}\right)I^\alpha_{d^-}\Lambda(c)\right) - \sigma m\left(\frac{x}{m} - d\right)^2\left(\frac{\Gamma(2\alpha+1)}{(d-c)^{2\alpha}} - \frac{\Gamma(\alpha+1)}{(d-c)^\alpha}\right)I^\alpha_{d^-}\Lambda(c)\Bigg]. \quad (3.15)$$

Proof. With the assumptions of P and Λ, we have

$$\frac{P(\Lambda(x)-\Lambda(c))}{\Lambda(x)-\Lambda(c)} \leq \frac{P(\Lambda(d)-\Lambda(c))}{\Lambda(d)-\Lambda(c)}. \quad (3.16)$$

Multiplying $T^{\eta,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,x}\left(\omega(\Lambda(x)-\Lambda(c))^\mu;\underline{a},\underline{b},\underline{e},s\right)\Lambda'(x)$ in equation (3.16) and from (2.22), we get

$$\begin{aligned}&D^c_x\left(T^{\eta,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,x}\Lambda;P\right)\Lambda'(x)\\ &\leq \frac{P(\Lambda(d)-\Lambda(c))}{\Lambda(d)-\Lambda(c)}T^{\eta,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,x}\left(\omega(\Lambda(x)-\Lambda(c))^\mu;\underline{a},\underline{b},\underline{e},s\right)\Lambda'(x).\end{aligned} \quad (3.17)$$

By using

$$\begin{aligned}&T^{\eta,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,x}\left(\omega(\Lambda(x)-\Lambda(c))^\mu;\underline{a},\underline{b},\underline{e},s\right)\\ &\leq T^{\eta,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,x}\left(\omega(\Lambda(d)-\Lambda(c))^\mu;\underline{a},\underline{b},\underline{e},s\right),\end{aligned} \quad (3.18)$$

we get the following inequality:

$$D^c_x(T^{\eta,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,x}\Lambda;P)\Lambda'(x) \leq D^c_d\left(T^{\eta,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,x}\Lambda;P\right)\Lambda'(x). \quad (3.19)$$

Using the definition of strongly (α, m)-convex function Υ for $x \in [c,d]$, the following inequality holds:

$$\begin{aligned}\Upsilon(x) \quad &\leq (\frac{x-c}{d-c})^\alpha\Upsilon(d) + m(1-\left(\frac{x-c}{d-c}\right)^\alpha)\Upsilon\left(\frac{c}{m}\right)\\ &-\sigma m(\frac{x-c}{d-c})^\alpha(1-\left(\frac{x-c}{d-c}\right)^\alpha)(\frac{c}{m}-d)^2.\end{aligned} \quad (3.20)$$

Multiplying (3.19) and (3.20) and integrating over $[c,d]$, we get

$$\begin{aligned}&\int_c^d D^c_x\left(T^{\eta,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,x}\Lambda;P\right)\Upsilon(x)d(\Lambda(x))\\ &\leq D^c_d\left(T^{\eta,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,x}\Lambda;P\right)[m\Upsilon\left(\frac{c}{m}\right)\int_c^d(1-\left(\frac{x-c}{d-c}\right)^\alpha)d(\Lambda(x))\\ &+\Upsilon(d)\int_c^d(\frac{x-c}{d-c})^\alpha d(\Lambda(x)) - \sigma m(\frac{c}{m}-d)^2\\ &\times\left(\int_c^d(\frac{x-c}{d-c})^\alpha d(\Lambda(x) - \int_c^d(\frac{x-c}{d-c})^{2\alpha}d(\Lambda(x))\right)].\end{aligned}$$

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We use Definition 10 and integrating by parts, then we get the following inequalities:

$$\begin{aligned}&({}_{\Lambda}^{P}A_{d^{-},\alpha,\beta,\gamma,\mu,x}^{\omega,\eta,\rho,\theta,k,r}\Upsilon)(c;s)\\&\le D_{d}^{c}\left(T_{\alpha,\beta,\gamma,\mu,x}^{\eta,\rho,\theta,k,r}\Lambda;P\right)[\left(\Upsilon(d)\Lambda(d)-m\Upsilon\left(\frac{c}{m}\right)\Lambda(c)\right)-\frac{\Gamma(\alpha+1)}{(d-c)^{\alpha}}\\&\times\left(\left(\Upsilon(d)-m\Upsilon\left(\frac{c}{m}\right)\right)I_{d^{-}}^{\alpha}\Lambda(c)\right)-\sigma m(\frac{c}{m}-d)^{2}(\frac{\Gamma(2\alpha+1)}{(d-c)^{2\alpha}}\\&-\frac{\Gamma(\alpha+1)}{(d-c)^{\alpha}})I_{d^{-}}^{\alpha}\Lambda(c)].\end{aligned}\tag{3.21}$$

Now, for $x\in[c,d]$, the following inequality holds true.

$$D_{d}^{x}\left(T_{\alpha,\beta,\gamma,\mu,x}^{\eta,\rho,\theta,k,r}\Lambda;P\right)\Lambda'(x)\le\Lambda_{d}^{c}\left(T_{\alpha,\beta,\gamma,\mu,x}^{\eta,\rho,\theta,k,r}\Lambda;P\right)\Lambda'(x).\tag{3.22}$$

Now, applying the similar process as we did for (3.19) and (3.20), the following inequality can be obtained from (3.20) and (3.22)

$$\begin{aligned}&({}_{\Lambda}^{P}A_{c^{+},\alpha,\beta,\gamma,\mu,x}^{\omega,\eta,\rho,\theta,k,r}\Upsilon)(d;s)\\&\le D_{d}^{c}\left(T_{\alpha,\beta,\gamma,\mu,x}^{\eta,\rho,\theta,k,r}\Lambda;P\right)[\left(\Upsilon(d)\Lambda(d)-m\Upsilon\left(\frac{c}{m}\right)\Lambda(c)\right)-\frac{\Gamma(\alpha+1)}{(d-c)^{\alpha}}\\&\times\left(\left(\Upsilon(d)-m\Upsilon\left(\frac{c}{m}\right)\right)I_{d^{-}}^{\alpha}\Lambda(c)\right)-\sigma m(\frac{c}{m}-d)^{2}(\frac{\Gamma(2\alpha+1)}{(d-c)^{2\alpha}}\\&-\frac{\Gamma(\alpha+1)}{(d-c)^{\alpha}})I_{d^{-}}^{\alpha}\Lambda(c)].\end{aligned}\tag{3.23}$$

By adding (3.21) and (3.23), the following inequality we get:

$$\begin{aligned}&({}_{\Lambda}^{P}A_{d^{-},\alpha,\beta,\gamma,\mu,x}^{\omega,\eta,\rho,\theta,k,r}\Upsilon)(c;s)+({}_{\Lambda}^{P}A_{c^{+},\alpha,\beta,\gamma,\mu,x}^{\omega,\eta,\rho,\theta,k,r}\Upsilon)(d;s))\\&\le 2D_{d}^{c}\left(T_{\alpha,\beta,\gamma,\mu,x}^{\eta,\rho,\theta,k,r}\Lambda;P\right)[(\Upsilon(d)\Lambda(d)\\&-m\Upsilon\left(\frac{c}{m}\right)\Lambda(c))-\frac{\Gamma(\alpha+1)}{(d-c)^{\alpha}}\left(\left(\Upsilon(d)-m\Upsilon\left(\frac{c}{m}\right)\right)I_{d^{-}}^{\alpha}\Lambda(c)\right)\\&-\sigma m\left(\frac{c}{m}-d\right)^{2}\left(\frac{\Gamma(2\alpha+1)}{(d-c)^{2\alpha}}-\frac{\Gamma(\alpha+1)}{(d-c)^{\alpha}}\right)I_{d^{-}}^{\alpha}\Lambda(c)\Big].\end{aligned}\tag{3.24}$$

Multiplying both sides of (2.25) by $D_{x}^{c}\left(T_{\alpha,\beta,\gamma,\mu,x}^{\eta,\rho,\theta,k,r}\Lambda;P\right)d\big(\Lambda(x)\big)$, and integrating over $[c,d]$, we have

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$$\begin{aligned}
&\Upsilon\left(\frac{c+md}{2}\right)\int_c^d D_x^c\left(T_{\alpha,\beta,\gamma,\mu,x}^{\eta,\rho,\theta,k,r}\Lambda;P\right)d\big(\Lambda(x)\big)\\
&\leq\left(\frac{1}{2^\alpha}+m\left(1-\frac{1}{2^\alpha}\right)\right)\int_c^d D_x^c\left(T_{\alpha,\beta,\gamma,\mu,x}^{\eta,\rho,\theta,k,r}\Lambda;P\right)\\
&\ \times d\big(\Lambda(x)\big)-\frac{\sigma}{m}\left(\frac{2^\alpha-1}{2^{2\alpha}}\right)\int_c^d D_x^c\left(T_{\alpha,\beta,\gamma,\mu,x}^{\eta,\rho,\theta,k,r}\Lambda;P\right)\\
&\ \times(c+md-x-mx)^2 d\big(\Lambda(x)\big).
\end{aligned}\tag{3.25}$$

From Definition 10, we get the required inequality:

$$\begin{aligned}
&\Upsilon\left(\frac{c+md}{2}\right)({}_{\Lambda}^{P}A_{d^-,\alpha,\beta,\gamma,\mu,x}^{\omega,\eta,\rho,\theta,k,r}1)(c;s)\\
&+\frac{\sigma}{m}\left(\frac{2^\alpha-1}{2^{2\alpha}}\right)({}_{\Lambda}^{P}A_{d^-,\alpha,\beta,\gamma,\mu,x}^{\omega,\eta,\rho,\theta,k,r}(c+md-x-mx)^2)(c;s)\\
&\leq\left(\frac{1+m(2^\alpha-1)}{2^\alpha}\right)({}_{\Lambda}^{P}A_{d^-,\alpha,\beta,\gamma,\mu,x}^{\omega,\eta,\rho,\theta,k,r}\Upsilon)(c;s).
\end{aligned}\tag{3.26}$$

Similarly, multiplying both sides of (2.25) by $D_d^x\left(T_{\alpha,\beta,\gamma,\mu,v}^{\eta,\rho,\theta,k,r}\Lambda;P\right)d\big(\Lambda(x)\big)$ and integrating over $[c,d]$, we have

$$\begin{aligned}
&\Upsilon\left(\frac{c+md}{2}\right)({}_{\Lambda}^{P}A_{c^+,\alpha,\beta,\gamma,\mu,x}^{\omega,\eta,\rho,\theta,k,r}1)(d;s)\\
&+\frac{\sigma}{m}\left(\frac{2^\alpha-1}{2^{2\alpha}}\right)({}_{\Lambda}^{P}A_{c^+,\alpha,\beta,\gamma,\mu,x}^{\omega,\eta,\rho,\theta,k,r}(c+md-x-mx)^2)(d;s)\\
&\leq\left(\frac{1+m(2^\alpha-1)}{2^\alpha}\right)({}_{\Lambda}^{P}A_{c^+,\alpha,\beta,\gamma,\mu,x}^{\omega,\eta,\rho,\theta,k,r}\Upsilon)(d;s).
\end{aligned}\tag{3.27}$$

By adding (3.26) and (3.27), the following inequality is obtained:

$$\begin{aligned}
&\frac{2^\alpha}{1+m(2^\alpha-1)}[\Upsilon\left(\frac{c+md}{2}\right)(({}_{\Lambda}^{P}A_{d^-,\alpha,\beta,\gamma,\mu,x}^{\omega,\eta,\rho,\theta,k,r}1)(c;s)\\
&\quad+({}_{\Lambda}^{P}A_{c^+,\alpha,\beta,\gamma,\mu,x}^{\omega,\eta,\rho,\theta,k,r}1)(d;s))\\
&\quad+\frac{\sigma}{m}\left(\frac{2^\alpha-1}{2^{2\alpha}}\right)(({}_{\Lambda}^{P}A_{d^-,\alpha,\beta,\gamma,\mu,x}^{\omega,\eta,\rho,\theta,k,r}(c+md-x-mx)^2)(c;s)\\
&\quad+({}_{\Lambda}^{P}A_{c^+,\alpha,\beta,\gamma,\mu,x}^{\omega,\eta,\rho,\theta,k,r}(c+md-x-mx)^2)(d;s))]\\
&\leq({}_{\Lambda}^{P}A_{d^-,\alpha,\beta,\gamma,\mu,x}^{\omega,\eta,\rho,\theta,k,r}\Upsilon)(c;s)+({}_{\Lambda}^{P}A_{c^+,\alpha,\beta,\gamma,\mu,x}^{\omega,\eta,\rho,\theta,k,r}\Upsilon)(d;s).
\end{aligned}\tag{3.28}$$

Combining (3.24) and (3.28), inequality (3.15) can be achieved. □

Corollary 3.4. *If w consider* $\alpha=1$, we get the following inequality satisfy for strongly m-convex *functions*

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$$\frac{2}{1+m}[\Upsilon\left(\frac{c+md}{2}\right)(({}^{P}_{\Lambda}A^{\omega,\eta,\rho,\theta,k,r}_{d^-,\alpha,\beta,\gamma,\mu,x}1)(c;s)+({}^{P}_{\Lambda}A^{\omega,\eta,\rho,\theta,k,r}_{c^+,\alpha,\beta,\gamma,\mu,x}1)(d;s))$$
$$+\frac{\sigma}{m}\left(\frac{1}{2^2}\right)(({}^{P}_{\Lambda}A^{\omega,\eta,\rho,\theta,k,r}_{d^-,\alpha,\beta,\gamma,\mu,x}(c+md-x-mx)^2)(c;s)$$
$$+({}^{P}_{\Lambda}A^{\omega,\eta,\rho,\theta,k,r}_{c^+,\alpha,\beta,\gamma,\mu,x}(c+md-x-mx)^2)(d;s))]$$
$$\leq({}^{P}_{\Lambda}A^{\omega,\eta,\rho,\theta,k,r}_{d^-,\alpha,\beta,\gamma,\mu,x}\Upsilon)(c;s)+({}^{P}_{\Lambda}A^{\omega,\eta,\rho,\theta,k,r}_{c^+,\alpha,\beta,\gamma,\mu,x}\Upsilon)(d;s)$$
$$\leq 2D^c_d\left(T^{\eta,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\mu,x}\Lambda;P\right)[\left(\Upsilon(d)\Lambda d-m\Upsilon\left(\frac{c}{m}\right)\Lambda(c)\right)$$
$$-\frac{1}{(d-c)}\left(\Upsilon(d)-m\Upsilon\left(\frac{c}{m}\right)\right)\int_c^d\Lambda(x)dx]$$
$$-\sigma m\left(\frac{c}{m}-d\right)^2\left(\frac{2(x-c)}{(d-c)^2}-\frac{1}{(d-c)}\right)\int_c^d\Lambda(x)dx\Big]. \tag{3.29}$$

Remark 6. *If $\alpha=1$ and $\sigma=0$, then above theorem reduces to Example 4 of [22].*

Remark 7. *If $(\alpha,m)=(1,1)$ and $\sigma=0$, then above theorem reduces to Example 5 of [22].*

Theorem 3.5. *Let $\Upsilon:[c,d]\to\Re$ be a differentiable function. If $|\Upsilon'|$ is strongly (α,m)-convex and we let $\Lambda:[c,d]\to\Re$ be a strictly increasing and differentiable function, then we also let $\frac{P}{x}$ be an increasing function on $[c,d]$. Then, for all $x\in[c,d]$, we have the following inequality involving the unified Mittag-Leffler function $T^{\eta,\rho,\theta,\kappa,r}_{\alpha,\beta,\gamma,\delta,\mu,x}\left(z;\underline{a},\underline{b},\mathrm{e},s\right)$ satisfying all the convergence conditions:*

$$\left|({}^{P}_{\Lambda}A^{\omega,\eta,\rho,\theta,k,r}_{c^+,\alpha,\beta,\gamma,\delta,\mu,x}\Upsilon*\Lambda)(x;s)+({}^{P}_{\Lambda}A^{\omega,\eta,\rho,\theta,k,r}_{d^-,\alpha,\beta,\gamma,\delta,\mu,x}\Upsilon*\Lambda)(x;s)\right|$$
$$\leq D^c_x\left(T^{\omega,\eta,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\delta,\mu,x}\Lambda;P\right)[\left(m\left|\Upsilon'\left(\frac{x}{m}\right)\right|\Lambda(x)-|\Upsilon'(c)|\Lambda(c)\right)$$
$$-\frac{\Gamma(\alpha+1)}{(x-c)^\alpha}\left(m\left|\Upsilon'\left(\frac{x}{m}\right)\right|-|\Upsilon'(c)|\right)I^\alpha_{C^+}\Lambda(x)-\sigma m(\frac{x}{m}-c)^2$$
$$\times\left(\frac{\Gamma(\alpha+1)}{(x-c)^\alpha}I^\alpha_{c^+}\Lambda(x)-\frac{\Gamma(2\alpha+1)}{(x-c)^{2\alpha}}I^{2\alpha}_{c^+}\Lambda(x)\right)]$$
$$+D^x_d\left(T^{\omega,\eta,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\delta,\mu,x}\Lambda;P\right)[\left(|\Upsilon'(d)|\Lambda(d)-m\left|\Upsilon'\left(\frac{x}{m}\right)\right|\Lambda(x)\right)$$
$$-\frac{\Gamma(\alpha+1)}{(d-x)^\alpha}\left(|\Upsilon'(d)|-m\left|\Upsilon'\left(\frac{x}{m}\right)\right|\right)I^\alpha_{d^-}\Lambda(x))-\sigma m(\frac{x}{m}-d)^2$$
$$\times\left(\frac{\Gamma(\alpha+1)}{(d-x)^\alpha}I^\alpha_{d^-}\Lambda(x)-\frac{\Gamma(2\alpha+1)}{(d-x)^{2\alpha}}I^{2\alpha}_{d^-}\Lambda(x)\right)], \tag{3.30}$$

where $({}^{P}_{\Lambda}A^{\omega,\eta,\rho,\theta,k,r}_{c^+,\alpha,\beta,\gamma,\delta,\mu,x}\Upsilon*\Lambda)(x;s):=\int_c^x D^\xi_x\left(T^{\eta,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\delta,\mu,x}\Lambda;P\right)\Upsilon'(\xi)d\big(\Lambda(\xi)\big)$

and $({}^{P}_{\Lambda}A^{\omega,\eta,\rho,\theta,k,r}_{d^-,\alpha,\beta,\gamma,\delta,\mu,x}\Upsilon*\Lambda)(x;s):=\int_x^d D^x_\xi\left(T^{\eta,\rho,\theta,k,r}_{\alpha,\beta,\gamma,\delta,\mu,x}\Lambda;P\right)\Upsilon'(\xi)d\big(\Lambda(\xi)\big).$

Proof. Let $x\in[c,d]$ and $\xi\in[c,x]$. Then using the definition of strongly (α,m)-convexity for $|\Upsilon'|$ the following inequality is valid:

$$\begin{aligned}|\Upsilon'(\xi)| \quad &\leq (\frac{x-\xi}{x-c})^{\alpha}|\Upsilon'(c)| + m(1-\left(\frac{x-\xi}{x-c}\right)^{\alpha})\left|\Upsilon'(\frac{x}{m})\right| \\ &-\sigma m(\frac{x-\xi}{x-c})^{\alpha}(1-\left(\frac{x-\xi}{x-c}\right)^{\alpha})(\frac{x}{m}-c)^2. \end{aligned} \tag{3.31}$$

The inequality (3.31) can be written as follows:

$$\begin{aligned}&-\Big[(\frac{x-\xi}{x-c})^{\alpha}|\Upsilon'(c)| + m(1-(\frac{x-\xi}{x-c})^{\alpha})\left|\Upsilon'\left(\frac{x}{m}\right)\right| \\ &-\sigma m(\frac{x-\xi}{x-c})^{\alpha}(1-(\frac{x-\xi}{x-c})^{\alpha})(\frac{x}{m}-c)^2\Big] \\ &\leq \Upsilon'(\xi) \\ &\leq (\frac{x-\xi}{x-c})^{\alpha}|\Upsilon'(c)| + m(1-\left(\frac{x-\xi}{x-c}\right)^{\alpha})\left|\Upsilon'(\frac{x}{m})\right| \\ &-\sigma m(\frac{x-\xi}{x-c})^{\alpha}(1-\left(\frac{x-\xi}{x-c}\right)^{\alpha})(\frac{x}{m}-c)^2. \end{aligned} \tag{3.32}$$

Now consider the second inequality of (3.32):

$$\begin{aligned}\Upsilon'(\xi) \quad &\leq (\frac{x-\xi}{x-c})^{\alpha}|\Upsilon'(c)| + m(1-\left(\frac{x-\xi}{x-c}\right)^{\alpha})\left|\Upsilon'(\frac{x}{m})\right| \\ &-\sigma m(\frac{x-\xi}{x-c})^{\alpha}(1-\left(\frac{x-\xi}{x-c}\right)^{\alpha})(\frac{x}{m}-c)^2. \end{aligned} \tag{3.33}$$

Multiplying (3.7) and (3.33) and integrating over $[c, x]$, we get

$$\begin{aligned}&\int_c^x D_x^{\xi}\left(T_{\alpha,\beta,\gamma,\delta,\mu,x}^{\eta,\rho,\theta,k,r}\Lambda; P\right)\Upsilon'(\xi)d\big(\Lambda(\xi)\big) \\ &\leq D_x^{c}\left(T_{\alpha,\beta,\gamma,\delta,\mu,x}^{\eta,\rho,\theta,k,r}\Lambda; P\right)[|\Upsilon'(c)|\int_c^x (\frac{x-\xi}{x-c})^{\alpha}d\big(\Lambda(\xi)\big) \\ &+m\left|\Upsilon'(\frac{x}{m})\right|\int_c^x (1-\left(\frac{x-\xi}{x-c}\right)^{\alpha})d\big(\Lambda(\xi)\big) \\ &-\sigma m(\frac{x}{m}-c)^2(\int_c^x (\frac{x-\xi}{x-c})^{\alpha}d\big(\Lambda(\xi)\big) \\ &-\int_c^x (\frac{x-\xi}{x-c})^{2\alpha}d\big(\Lambda(\xi)\big))]. \end{aligned}$$

By applying (2.20) of Definition 10 and integrating by parts, we get

$$\begin{aligned}
&({}_{\Lambda}^{P}A_{c^+,\alpha,\beta,\gamma,\delta,\mu,x}^{\omega,\eta,\rho,\theta,k,r}\Upsilon * \Lambda)(x;s)\\
&\leq D_x^c\left(T_{\alpha,\beta,\gamma,\delta,\mu,x}^{\omega,\eta,\rho,\theta,k,r}\Lambda;P\right)[\left(m\left|\Upsilon'(\frac{x}{m})\right|\Lambda(x)-|\Upsilon'(c)|\Lambda(c)\right)\\
&-\frac{\Gamma(\alpha+1)}{(x-c)^\alpha}\left(m\left|\Upsilon'(\frac{x}{m})\right|-|\Upsilon'(c)|\right)I_{c^+}^{\alpha}\Lambda(x)-\sigma m(\frac{x}{m}-c)^2\\
&\times\left(\frac{\Gamma(\alpha+1)}{(x-c)^\alpha}I_{c^+}^{\alpha}\Lambda(x)-\frac{\Gamma(2\alpha+1)}{(x-c)^{2\alpha}}I_{c^+}^{2\alpha}\Lambda(x)\right)]. \qquad (3.34)
\end{aligned}$$

If we take the left-hand side of the inequality (3.32) and follow the similar pattern as for the right-hand side inequality, then

$$\begin{aligned}
&({}_{\Lambda}^{P}A_{c^+,\alpha,\beta,\gamma,\delta,\mu,x}^{\omega,\eta,\rho,\theta,k,r}\Upsilon * \Lambda)(x;s)\\
&\geq -D_x^c\left(T_{\alpha,\beta,\gamma,\delta,\mu,x}^{\omega,\eta,\rho,\theta,k,r}\Lambda;P\right)[\left(m\left|\Upsilon'(\frac{x}{m})\right|\Lambda(x)-|\Upsilon'(c)|\Lambda(c)\right)\\
&-\frac{\Gamma(\alpha+1)}{(x-c)^\alpha}\left(m\left|\Upsilon'(\frac{x}{m})\right|-|\Upsilon'(c)|\right)I_{C^+}^{\alpha}\Lambda(x)-\sigma m(\frac{x}{m}-c)^2\\
&\times\left(\frac{\Gamma(\alpha+1)}{(x-c)^\alpha}I_{c^+}^{\alpha}\Lambda(x)-\frac{\Gamma(2\alpha+1)}{(x-c)^{2\alpha}}I_{c^+}^{2\alpha}\Lambda(x)\right)]. \qquad (3.35)
\end{aligned}$$

From (3.34) and (3.35), the following inequality is observed:

$$\begin{aligned}
&\left|({}_{\Lambda}^{P}A_{c^+,\alpha,\beta,\gamma,\delta,\mu,x}^{\omega,\eta,\rho,\theta,k,r}\Upsilon * \Lambda)(x;s)\right|\\
&\leq D_x^c\left(T_{\alpha,\beta,\gamma,\delta,\mu,x}^{\omega,\eta,\rho,\theta,k,r}\Lambda;P\right)[(m|\Upsilon'\left(\frac{x}{m}\right)|\Lambda(x)-|\Upsilon'(c)|\Lambda(c))\\
&-\frac{\Gamma(\alpha+1)}{(x-c)^\alpha}(m|\Upsilon'\left(\frac{x}{m}\right)|-|\Upsilon'(c)|)I_{C^+}^{\alpha}\Lambda(x)-\sigma m(\frac{x}{m}-c)^2\\
&\times\left(\frac{\Gamma(\alpha+1)}{(x-c)^\alpha}I_{c^+}^{\alpha}\Lambda(x)-\frac{\Gamma(2\alpha+1)}{(x-c)^{2\alpha}}I_{c^+}^{2\alpha}\Lambda(x)\right)]. \qquad (3.36)
\end{aligned}$$

Now, applying strongly (α,m)-convexity of $|\Upsilon'|$ on $(x,d]$ for $x\in[c,d]$ we have

$$\begin{aligned}
|\Upsilon'(\xi)| \quad &\leq \left(\frac{\xi-x}{d-x}\right)^\alpha|\Upsilon'(d)|+m\left(1-\left(\frac{\xi-x}{d-x}\right)^\alpha\right)\left|\Upsilon'\left(\frac{x}{m}\right)\right|\\
&-\sigma m(\frac{\xi-x}{d-x})^\alpha(1-\left(\frac{\xi-x}{d-x}\right)^\alpha\right)(\frac{x}{m}-d)^2. \qquad (3.37)
\end{aligned}$$

Applying the same procedure as we did for above, we get the following inequality from (3.11) and (3.37).

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$$\left|\left({}_{\Lambda}^{P}A_{d^-,\alpha,\beta,\gamma,\delta,\mu,x}^{\omega,\eta,\rho,\theta,k,r}\Upsilon * \Lambda\right)(x;s)\right|$$
$$\leq D_d^x\left(T_{\alpha,\beta,\gamma,\delta,\mu,x}^{\omega,\eta,\rho,\theta,k,r}\Lambda;P\right)\Big[(|\Upsilon'(d)|\Lambda(d) - m|\Upsilon'\left(\frac{x}{m}\right)\Big|\Lambda(x)\Big)$$
$$-\frac{\Gamma(\alpha+1)}{(d-x)^{\alpha}}(|\Upsilon'(d)| - m|\Upsilon'\left(\frac{x}{m}\right)|)I_{d^-}^{\alpha}\Lambda(x)) \tag{3.38}$$
$$-\sigma m\left(\frac{x}{m}-d\right))^2\left(\frac{\Gamma(\alpha+1)}{(d-x)^{\alpha}}I_{d^-}^{\alpha}\Lambda(x) - \frac{\Gamma(2\alpha+1)}{(d-x)^{2\alpha}}I_{d^-}^{2\alpha}\Lambda(x)\right)].$$

After adding (3.36) a and (3.38), inequality (3.30) can be obtained.

Corollary 3.6. *If $\alpha = 1$ then the following inequality holds for the strongly m-convex function*

$$\left|\left({}_{\Lambda}^{P}A_{c^+,\alpha,\beta,\gamma,\delta,\mu,x}^{\omega,\eta,\rho,\theta,k,r}\Upsilon * \Lambda\right)(x;s) + \left({}_{\Lambda}^{P}A_{d^-,\alpha,\beta,\gamma,\delta,\mu,x}^{\omega,\eta,\rho,\theta,k,r}\Upsilon * \Lambda\right)(x;s)\right|$$
$$\leq D_x^c\left(T_{\alpha,\beta,\gamma,\delta,\mu,x}^{\omega,\eta,\rho,\theta,k,r}\Lambda;P\right)[(m|\Upsilon'\left(\frac{x}{m}\right)|\Lambda(x) - |\Upsilon'(c)|\Lambda(c))$$
$$-\frac{1}{(x-c)}(m|\Upsilon'\left(\frac{x}{m}\right)| - |\Upsilon'(c)|)$$
$$\int_c^x \Lambda(\xi)d(\xi) - \sigma m(\frac{x}{m}-c)^2(\frac{1}{(x-c)}\int_c^x \Lambda(\xi)d(\xi)$$
$$-\frac{2}{(x-c)^2}\int_c^x (x-\xi)\Lambda(\xi)d(\xi))]$$
$$+D_d^x\left(T_{\alpha,\beta,\gamma,\delta,\mu,x}^{\omega,\eta,\rho,\theta,k,r}\Lambda;P\right)[(|\Upsilon'(d)|\Lambda(d) - m|\Upsilon'\left(\frac{x}{m}\right)|\Lambda(x))$$
$$-\frac{1}{(d-x)}(|\Upsilon'(d)| - m|\Upsilon'\left(\frac{x}{m}\right)|)$$
$$\int_x^d \Lambda(\xi)d(\xi) - \sigma m(\frac{x}{m}-d)^2(\frac{1}{(d-x)}\int_x^d \Lambda(\xi)d(\xi)$$
$$-\frac{2}{(d-x)^2}\int_x^d (\xi-x)\Lambda(\xi)d(\xi))].$$

Remark 8. *If $\alpha = 1$ and $\sigma = 0$, then above theorem reduces to Example 6 of [22] .*

Remark 9. *If $(\alpha, m) = (1,1)$ and $\sigma = 0$, then above theorem reduces to Example 7 of [22] .*

Remark 10. *If $\sigma = 0$, then above theorem reduces to Theorem 3 of [22].*

VI. CONCLUSIONS

In this article, we have established integral inequalities for strongly (α, m)-convex function by using fractional integral operators involving unified Mittag-Leffler function and also proved the H-H type inequality with the help of strongly (α, m)-convex functions close to symmetric functions. The results obtained in this paper contain several previous established results for m-convex and (α, m)-convex functions. As future directions we can extend the result of this paper for higher order strongly (α, m)-convex function.

DECLARATION

ETHICAL APPROVAL

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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