Computational Analysis of a queueing problem using the minimal and maximal transfer rules

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Abstract— **We study the implementation of the Shapley value using the minimal and maximal transfer rules, which serve as mechanisms to minimize and maximize waiting costs, respectively. We developed a computational algorithm to determine the minimal and maximal waiting costs for items and also presented the fair distribution of waiting costs in tabular form for both cases. We introduce a new step-wise linear incremental unit waiting cost system instead of constant unit waiting cost and apply it to a cold storage scenario, optimizing the management of perishable goods by prioritizing or delaying processing based on cost variations. We conclude that minimal and maximal transfer rules, with step-wise linear incremental unit waiting cost, provide a fair and more realistic cost distribution of items.**

Keywords— Queueing problems, Shapley value, Inventory management.

I. INTRODUCTION

In inventory management, the goal is to minimize costs related to holding, ordering, and stockouts to maintain efficiency and mitigate losses. In such systems, items are queued to optimize processing, replenishment, and dispatching, ensuring cost efficiency. We focus on prioritizing inventory based on perishability or cost, where highly perishable items are queued first to prevent spoilage, while items with lower turnover rates are queued later.

The rules derived from cooperative game theory for queueing problems can effectively be used to address the challenge of allocating resources and prioritizing actions when dealing with multiple inventory demands. Maniquet's (2003) [1], considers an optimistic approach. The author assumes that coalition members are served before those who are not part of the coalition. Chun (2008) [4], suggests a pessimistic approach. So, in contrast, coalition members are assumed to be served after non-coalitional members. This comparison highlights how the assumptions about service orders can significantly affect the outcomes of queueing problems. In both approaches to queueing problems, the Shapley value (1953) [3], a fundamental concept in cooperative game theory, is utilized to fairly distribute costs among agents.

II. PRELIMINARIES

In a queueing situation, a set of items $N = \{1,2,\dots,n\}$ faces the problem of dividing the total

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waiting costs associated with being served at a facility. Each item incurs a cost depending on its position in the queue, and the objective is to fairly allocate these costs among the items based on their respective waiting times and the order in which they are served. Let $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$ represent the queue order, where σ_i denote the position of agent i in the queue. Each item i has an associated unit waiting cost θ_i assumed to be constant per unit of time. Define $F_i(\sigma)$ as the set of items following agent i in the queue, and $P_i(\sigma)$ as the set of items preceding agent i in the queue. For each item, the unit waiting cost differs. $|N|$ is the number of items in the grand coalition. $|S|$ is the number of items in the coalition S. In the optimistic approach [1], the worth of a coalition S is defined by assuming that all items of the coalition S are served before the items of $N\backslash S$. The worth $v_M(S)$ of the coalition S under this assumption is given by:

$$v_M(S) = -\sum_{i\in S}(\sigma_i - 1)\theta_i \tag{1}$$

where coalition S items are served first. The value depends on the waiting costs assuming the coalition items are prioritized. The Shapley value is then applied to this game to determine the monetary transfers. In the pessimistic approach, the worth of a coalition S is defined by assuming that all items of the coalition S are served after the items of $N\backslash S$. The worth $v_P(S)$ of the coalition S under this assumption is given by:

$$v_P(S) = -\sum_{i\in S}(|N| - |S| + \sigma_i - 1)\theta_i \tag{2}$$

where coalition S items are served last. The value is adjusted by the total number of agents minus the coalition size, reflecting the additional waiting time imposed by serving all other items first.

The Shapley value $S_{pi}(v)$ for a player i in a cooperative game with characteristic function v is given by:

$$S_{pi}(v) = \sum_{S\subseteq N\backslash\{i\}} \frac{|S|.(|N|-|S|-1)!}{|N|!}[v(S\cup\{i\}) - v(S)] \tag{3}$$

Because the two approaches define coalition worth differently, the application of the Shapley value to each game results in different compensation rules for the items. In the optimistic approach, items in a coalition are seen as incurring minimal waiting costs, while in the pessimistic approach, they incur maximum waiting costs due to being served later.

III. ALGORITHM

Input:

- Number of items = n
- Unit waiting cost, $\theta = (\theta_1, \theta_2, \dots, \theta_n)$

Output:

- Transfers $t_{min} = (t_{1,min}, t_{2,min}, \dots, t_{n,min})$ and utilities $u_{min} = (u_{1,min}, u_{2,min}, \dots, u_{n,min})$ using the Minimal Transfer Rule
- Transfers $t_{max} = (t_{1,max}, t_{2,max}, \dots, t_{n,max})$ and utilities $u_{max} = (u_{1,max}, u_{2,max}, \dots, u_{n,max})$ using the Maximal Transfer Rule

Step 1

- Define the set of all items, $\{1,2, \dots, n\}$ and i denote any item of the set.

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- Create arrays t_{min}, t_{max} and u_{min}, u_{max} of size n.
- Initialize all elements of t and u to zero:

$$t_{i,min} \leftarrow 0, \qquad t_{i,max} \leftarrow 0$$
$$u_{i,min} \leftarrow 0, \qquad u_{i,max} \leftarrow 0$$

Step 2

- Sort the items in increasing order of their unit waiting costs $\theta_i: agent \leftarrow sort(agent, by\ \theta_i)$
- For each item i in the sorted order:
- Determine the set $F_i(\sigma)$ of items following item $i: F_i(\sigma) = \{\sigma_j \mid j > \sigma_i\}$
- Compute the sum of waiting costs for items in $F_i(\sigma): sum\_theta\_j = \sum_{j \in F_i(\sigma)} \theta_j$
- Compute the transfer $t_{i,min}$ for item $i: t_{i,min} = \frac{(\sigma_i - 1)\theta_i}{2} - \frac{1}{2}\sum_{j \in F_i(\sigma)} \theta_j$
- Calculate the utility $u_{i,min}$ for item $i: u_{i,min} = t_{i,min} - (\sigma_i - 1)\theta_i$

Step 3

- Sort the items in decreasing order of their waiting costs $\theta_i: agent \leftarrow sort(agent, by - \theta_i)$
- For each agent i in the sorted order:
- Determine the set $P_i(\sigma)$ of items preceding item $i: P_i(\sigma) = \{\sigma_j \mid j < \sigma_i\}$
- Compute the sum of waiting costs for agents in $P_i(\sigma): sum\_theta\_j = \sum_{j \in P(\sigma)} \theta_j$
- Calculate the transfer $t_{i,max}$ for agent $i: t_{i,max} = \frac{1}{2}\sum_{j \in P_i(\sigma)} \theta_j - \frac{(\sigma_i - 1)\theta_i}{2}$
- Calculate the utility $u_{i,max}$ for agent $i: u_{i,max} = \frac{1}{2}\sum_{j \in P_i(\sigma)} \theta_j - \frac{(n + \sigma_i - 2)\theta_i}{2}$

Output:

- Arrays t_{min} and u_{min} containing the transfer amounts and utilities computed using the Minimal Transfer Rule.
- Arrays t_{max} and u_{max} containing the transfer amounts and utilities computed using the Maximal Transfer Rule.

VI. A QUEUEING PROBLEM

We consider an example of queueing five items, each with a constant unit waiting cost. Following Maniquet's minimal transfer rule and Chun's maximal transfer rule, we determine the efficient queue order that minimizes and maximizes the total waiting cost.

A C program has been developed to implement the combined algorithm for computing transfers and utilities. This program calculates transfers and utilities using the Minimal and Maximal Transfer Rules, where agents are sorted in increasing order of unit waiting costs for the Minimal Rule and decreasing order for the Maximal Rule.

UNIT WAITING COST FOR EACH ITEM

Item	Unit Waiting Cost (θ_i)
a	2
b	3
c	1
d	4
e	2.5

A. *Calculations of transfers and utilities using the minimal transfer rules (Optimistic approach).*

TRANSFERS AND UTILITIES

Item	Position	Waiting Cost (W_i)	Transfer (t_i)	Utility (u_i)
c	1	0	-5.75	-5.75
a	2	2	-3.75	-5.75
e	3	5	-1	-6
b	4	9	2.5	-6.5
d	5	16	8	-8
Total		**32**	**0**	**-32**

B. *Calculation of transfers and utilities using the maximal transfer rule (Pessimistic approach).*

TRANSFERS AND UTILITIES

Item	Position	Waiting Cost (W_i)	Transfer (t_i)	Utility (u_i)
d	1	16	-8	-8
b	2	15	-2.5	-5.50
e	3	15	1	-4
a	4	14	3.75	-2.25
e	5	8	5.75	1.75
Total		**68**	**0**	**-18**

C. *Calculation of worth and payoff*

WORTH OF COALITIONS

Coalition	Optimistic Worth	Pessimistic Worth
$\{a\}$	0	-8
$\{b\}$	0	-12

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$\{c\}$	0	-4
$\{d\}$	0	-16
$\{e\}$	0	-10
$\{a,b\}$	-3	-17
$\{a,c\}$	-2	-10
$\{a,d\}$	-4	-20
$\{a,e\}$	-2.5	-15.5
$\{b,c\}$	-3	-13
$\{b,d\}$	-4	-24
$\{b,e\}$	-3	-19
$\{c,d\}$	-4	-16
$\{c,e\}$	-2.5	-11.5
$\{d,e\}$	-4	-22
$\{a,b,c\}$	-8	-16
$\{a,b,d\}$	-11	-25
$\{a,b,e\}$	-8.5	-21.5
$\{a,c,d\}$	-10	-18
$\{a,c,e\}$	-7	-15
$\{a,d,e\}$	-10.5	-23.5
$\{b,c,d\}$	-11	-21
$\{b,c,e\}$	-8.5	-17.5
$\{b,d,e\}$	-11	-27
$\{c,d,e\}$	-10.5	-19.5
$\{a,b,c,d\}$	-20	-20
$\{a,b,c,e\}$	-16	-18
$\{a,b,d,e\}$	-20.5	25.5
$\{a,c,d,e\}$	-19	-19
$\{b,c,d,e\}$	-20.5	-21.5
$\{a,b,c,d,e\}$	-32	-18

SHAPLEY VALUE PAYOFFS

Items	$S_{pi}(v_M, N)$	$S_{pi}(v_P, N)$
a	-5.75	-2.25
b	-6.50	-5.50
c	-5.75	1.75

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d	-8	-8
e	-6	-4
Total	**-32**	**-18**

Remark: The comparative analysis of coalition worth and Shapley value payoffs under optimistic and pessimistic approaches highlights the significance of service order on cost distribution in queueing problems. In the optimistic scenario, serving coalition members first results in higher cumulative costs and a total Shapley payoff of -32. In contrast, the pessimistic scenario, where coalition members are served last, leads to a lower total Shapley payoff of -18, indicating a more balanced cost distribution. Computation shows that the utilities calculated using the minimal and maximal transfer rules align with the Shapley value payoffs of the agents.

VII. Piecewise Linear Waiting Costs on Queue Management

In many real-world situations, costs do not increase uniformly over time. We consider the unit waiting cost to be stepwise linear incremental and apply the maximal transfer rule to order the items in the queue.

A. *Example:* We consider a scenario with three batches of apples stored in a cold storage facility using Controlled Atmosphere (CA) technology to extend their shelf life. Initially, the apples ripen naturally, causing a moderate increase in unit waiting costs. Once CA conditions are fully optimized, the ripening process slows, leading to a decrease in the deterioration rate and a reduction in waiting costs. However, as storage time continues, the apples eventually begin to over-ripen, resulting in a rise in waiting costs. In such a situation, a storage facility can be effectively managed by using the maximal transfer rule.

B. *Maximal Transfer Rule Calculation for All Intervals*

We consider a stepwise linear incremental function where the unit waiting cost increases linearly in steps but is not continuous, we can define the function $\theta_i(I)$ which represents the unit waiting cost for item i at a given interval I as follows:

$$\theta_i(I) = \begin{cases} \theta_{i,0} & if\ I_0 \le I \le I_{i,1} \\ \theta_{i,1} = \theta_{i,0} + \Delta_{i,1} & if\ I_{i,1} \le I \le I_{i,2} \\ \theta_{i,2} = \theta_{i,1} + \Delta_{i,2} & if\ I_{i,2} \le I \le I_{i,3} \\ \vdots & \\ \theta_{i,k} = \theta_{i,k-1} + \Delta_{i,k} & if\ I_{i,k-1} \le I \le I_{i,k} \end{cases} \tag{4}$$

Where:

- $I_0 = 0$ is the initial time when the item enters storage.
- $I_{i,1}, I_{i,2}, \ldots, I_{i,k}$ are the time intervals specific to item i after which the unit waiting cost

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changes.

- $\theta_{i,0}$ is the initial unit waiting cost for item i.
- $\Delta_{i,1}, \Delta_{i,2}, \ldots, \Delta_{i,k}$ represent the incremental changes in cost at each interval for item i.

The following function is the case of three batches of apples $\{a, b, c\}$ with step-wise linear incremental unit waiting cost:

$$\theta_i(I) = \begin{cases} (2,3,1) \;\; if \;\; I_1 \leq I < I_2 \\ (3,2,4) \;\; if \;\; I_2 \leq I < I_3 \\ \;\;(4,1,5) \quad if \;\; I_3 \leq I \end{cases} \tag{5}$$

C. Transfers and Utility

TRANSFERS AND UTILITIES FOR INTERVALS

Interval	t_a	t_b	t_c	u_a	u_b	u_c
I_1	0.5	-3	2.5	-0.5	-3	0.5
I_2	0.5	3.5	-4	-1	-0.5	-4
I_3	0.5	4.5	-5	-1.5	2.5	-5
Total	1.5	5	6.5	-3	-1	-8.5

D. Worth Calculation

WORTHS $v_M(S)$ OF EACH COALITION IN ALL INTERVALS

Coalition	I_1	I_2	I_3
$\{a\}$	-4	-6	-8
$\{b\}$	-6	-4	-2
$\{c\}$	-2	-8	-10
$\{a, b\}$	-6	-4	-5
$\{c, a\}$	-4	-7	-9
$\{c, b\}$	-4	-6	-6
$\{c, a, b\}$	-12	-18	-20

SHAPLEY VALUE PAYOFFS

Item	$I_1 = S_{pi}(v_M, N)$	$I_2 = S_{pi}(v_M, N)$	$I_3 = S_{pi}(v_M, N)$

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a	-4.33	-5.83	-7.66
b	-5.33	-4.33	-3.16
c	-2.33	-7.83	-9.16
Total	**-12**	**-18**	**-20**

This model allows the storage facility to optimize storage strategies by considering the unique deterioration profiles of each batch. For a certain batch, the facility could delay processing during the period of decreased deterioration, effectively taking advantage of the lower costs, before prioritizing the batch as deterioration increases again.

CONCLUSION

The above comparative analysis highlights that the order of service significantly impacts cost distribution in queueing systems. The proposed model with piecewise linear incremental unit waiting costs effectively captures the non-uniform cost increases over time, as demonstrated by the management of three batches of apples in cold storage. We conclude that the maximal transfer rule and the Shapley value payoffs enable us to make strategic decisions on ordering the batches by altering waiting costs in different intervals. The future scope of research is to develop a robust ordering method to minimize the waiting cost and prioritize perishable items.

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