Transient Analysis of a Disaster Queue with Balking and Optional Vacation

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***Abstract*— The paper considers a disastrous queue with balking and optional vacation. The study is of transient nature. We have obtained explicit expressions for time-variable system probabilities as well as effectiveness measures like mean and variance. All the analysis has been done by using probability generating function, iterative procedures, modified Bessel's function of first kind and Laplace transform.**

Key words: Transient analysis, disasters, optional vacation, balking.

I. INTRODUCTION

Disasters are real-time phenomena which renders a system to a complete stoppage of its services. Sources of disasters are various, e.g., natural (disruption of communication network due to bad weather), system overload (heavy power demand in peak hours), cyber-attacks in computer systems etc. Queueing researchers included this real-time happening phenomena into their research list in modelling various real-time systems that are capable of modelled as a queueing system. Other words that are used in place of disaster are- catastrophe, queue flushing, stochastic clearing etc. [1].

Vacationing server is an impotant practical dimension in the study of queueing systems (see the survey paper by [2]). Some papers that discussed disaters together with vacations are, e.g., [3] (transient analysis), [4] (transient and steady-state analysis), [5] (steady state analysis), [6] (steady state analysis). Visible effect of disater and vacation that one may see is impatience in customer either in the form of balking and/or in the form of reneging. For recent works on disaster queues incorporating impatience (together with other specialities), we may mention, e.g., [7] (steady-state analysis), [8] (steady-state analysis), [9].

Reference [5], mentioned in above literature review, introduced an optional vacation policy for the first time and studied a solo server disaster queue under this policy in the steady-state scenario; and obtained steady-state probabilities, performance measures together with cost analysis. But as far as we know, transient analysis and impatience behaviour are also equally important feature of a real system and this important dimension has not been explored till date under optional vacation policy. So we intend to fill this gap. Our objective in this paper is to update the study of [5] by presenting a trasient analysis of the same with additional realistic consideration of balking behaviour of customers during vacations.

Paper is organized in following manner: section II narrates the model and the symbols. Section III presents time-variable analysis. Performance measures are given in section IV. Conclusion, future scope and references are given in section V.

II. MODEL NARRATION

The model is solo server model with disaster, optional vacations and balking:

- Customers reach the server in Poisson fashion with rate λ.
- Customers are served with exponential rate μ.
- The system face disaters with Poisson rate θ in all of its states. Disaster renders the server incapable to serve as well as nudges out all the customers forever-- waiting or in-service. Crashed server is repaired immediately with exponential rate ξ and the server attains its operative state as soon as it is repaired.
- Server can avail two types of vacation- vacation 1 and vacation 2. After a busy cycle, when system is empty, the server can opt either forvacation 2 with probability p or for vacation 1 with probability q with exponential rate μp or μq respectively ($p+q=1$). During the vacation spells an arriving customer may join the queue with probability β or may balk with coplementary probability 1-β.
- Length of each vacation follows exponential distribution with rate δ. When server returns from vacation 1 and the system is still empty, the server goes in vacation 2.

Let ₦(t) be the total number of the customers (waiting plus in-service) in the system and $\mathscr{J}$(t) be the state of the system at the time 't'. $\mathscr{J}$(t)=1, 2, a and Q for vacation 1, vacation 2, busy and disaster states respectively. Then $\text{♀} = \{(n,i): n \geq 1, i = 1,2,a\} \cup \{Q,(0,1),(0,2)\}$ is the state space of continuous-time markov process CTMP {(₦(t), $\mathscr{J}$(t)), t ≥ 0}.

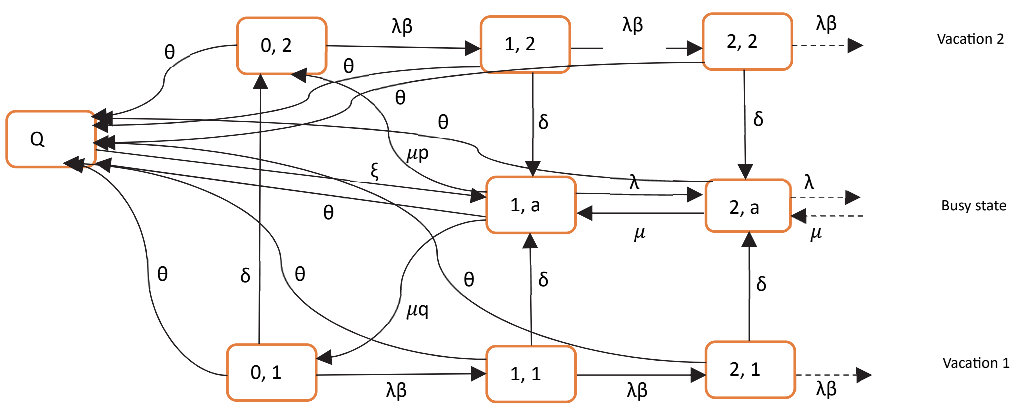

Fig.1. Display of state-transitions

Let $P_{n,i}(t)$ is the state probabilities of the CTMP {(Ꞥ(t), 𝒥(t)), t ≥ 0}. State transitions are shown in figure 1. Initially, we assume that the system is in disaster state Q so $Q(0) = 1$. Following are the balance equations:

Vacation 2.

$$P'_{0,2}(t) = -(\theta + \lambda\beta)P_{0,2}(t) + \mu p P_{1,a}(t) + \delta P_{0,1}(t) \quad (1)$$

$$P'_{l,2}(t) = -(\theta + \delta + \lambda\beta)P_{l,2}(t) + \lambda\beta P_{l-1,2}(t), \quad l \geq 1 \quad (2)$$

Vacation 1.

$$P'_{0,1}(t) = -(\theta + \delta + \lambda\beta)P_{0,1}(t) + \mu q P_{1,a}(t) \quad (3)$$

$$P'_{l,1}(t) = -(\theta + \delta + \lambda\beta)P_{l,1}(t) + \lambda\beta P_{l-1,1}(t), \quad l \geq 1 \quad (4)$$

Busy Period ‘a’.

$$P'_{1,a}(t) = -(\theta + \mu + \lambda)P_{1,a}(t) + \mu P_{2,a}(t) + \delta\left(P_{1,1}(t) + P_{1,2}(t)\right) + \xi Q(t) \quad (5)$$

$$P'_{l,a}(t) = -(\theta + \mu + \lambda)P_{l,a}(t) + \mu P_{l+1,a}(t) + \lambda P_{l-1,a}(t) + \delta\left(P_{l,1}(t) + P_{l,2}(t)\right), l \geq 2 \quad (6)$$

Disaster ‘Q’.

$$Q'(t) = -\xi Q(t) + \theta(1 - Q(t)) \quad (7)$$

III. TRANSIENT ANALYSIS

Equation (7) is a linear differential equation with integrating factor $e^{(\xi+\theta)t}$, so its solution is

$$Q(t) = \frac{\theta}{\theta + \xi} + \frac{\xi}{\theta + \xi} e^{-(\xi+\theta)t} \quad (8)$$

Let $\bar{P}_{l,i}(s)$ is the Laplace transform of $P_{l,i}(t)$. Taking Laplace transform of (1) through (5), we get

$$s\bar{P}_{0,2}(s) = -(\theta + \lambda\beta)\bar{P}_{0,2}(s) + \mu p \bar{P}_{1,a}(s) + \delta\bar{P}_{0,1}(s) \quad (9)$$

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$$s\bar{P}_{l,2}(s) = -(\theta+\delta+\lambda\beta)\bar{P}_{l,2}(s) + \lambda\beta\bar{P}_{l-1,2}(s)\,,\quad l \geq 1 \qquad (10)$$

$$s\bar{P}_{0,1}(s) = -(\theta+\delta+\lambda\beta)\bar{P}_{0,1}(s) + \mu q\bar{P}_{1,a}(s) \qquad (11)$$

$$s\bar{P}_{l,1}(s) = -(\theta+\delta+\lambda\beta)\bar{P}_{l,1}(s) + \lambda\beta\bar{P}_{l-1,1}(s) \qquad (12)$$

$$s\bar{P}_{1,a}(s) = -(\theta+\mu+\lambda)\bar{P}_{1,a}(s) + \mu\bar{P}_{2,a}(s) + \delta\left(\bar{P}_{1,1}(s)+\bar{P}_{1,2}(s)\right) + \xi\bar{Q}(s) \qquad (13)$$

From (10) and (12), we get, respectively $\bar{P}_{l,2}(s) = \left(\frac{\lambda\beta}{s+\theta+\delta+\lambda\beta}\right)^{l}\bar{P}_{0,2}(s), l \geq 1 \qquad (14)$

$$\bar{P}_{l,1}(s) = \left(\frac{\lambda\beta}{s+\theta+\delta+\lambda\beta}\right)^{l}\bar{P}_{0,1}(s), l \geq 1 \qquad (15)$$

Inverting (14) and (15), we get, respectively

$$P_{l,2}(t) = \frac{(\lambda\beta)^{l}}{(l-1)!}\,t^{l-1}e^{-(\theta+\delta+\lambda\beta)t} * P_{0,2}(t)\,, l \geq 1 \qquad (16)$$

$$P_{l,1}(t) = \frac{(\lambda\beta)^{l}}{(l-1)!}\,t^{l-1}e^{-(\theta+\delta+\lambda\beta)t} * P_{0,1}(t)\,, l \geq 1 \qquad (17)$$

To find the expression for $P_{l,a}(t)$, we suppose $\mathbb{P}(z,t) = \sum_{n=2}^{\infty} P_{n,a}(t)\,z^{n},\ \ \mathbb{P}(z,0) = 0$

Differentiating partially above equation with respect to t, using (6) and some algebra, we get

$$\frac{\partial\mathbb{P}(z,t)}{\partial t} - \left\{-(\theta+\lambda+\mu) + \left(\frac{\mu}{z}+\lambda z\right)\right\}\mathbb{P}(z,t) = \lambda z^{2}P_{1,a}(t) - \mu z P_{2,a}(t) + \mathbb{H}(t) \qquad (18)$$

where $\mathbb{H}(t) = \sum_{l=2}^{\infty}\delta\left(P_{l,1}(t)+P_{l,2}(t)\right)z^{l}$. Solving (18) by standard procedure, we get

$$\mathbb{P}(z,t) = \int_0^t\left[\lambda z^{2}P_{1,a}(y) - \mu z P_{2,a}(y) + \mathbb{H}(t)\right]e^{-(\theta+\lambda+\mu)(t-y)}e^{\left(\frac{\mu}{z}+\lambda z\right)(t-y)}dy \qquad (19)$$

Let $\alpha = 2\sqrt{\lambda\mu}\,,\ \gamma = \sqrt{\frac{\lambda}{\mu}}$, then $e^{\left(\frac{\mu}{z}+\lambda z\right)t} = \sum_{-\infty}^{\infty}(\gamma z)^{l}I_{l}(\alpha t)$, $I_l(.)$ is the modified Bessel's function of kind first. Using this identity in above equation and then comparing the coefficients of z^{l} and z^{-l}

on both the sides, we get two equations, which upon solving gives

$$P_{l,a}(t) = \lambda \int_0^t \gamma^{l-2} P_{1,a}(y) e^{-(\theta+\lambda+\mu)(t-y)} \left[I_{l-2}\big(\alpha(t-y)\big) - I_{l+2}\big(\alpha(t-y)\big)\right] dy$$
$$- \int_0^t \mu\gamma^{l-1} P_{2,a}(y) e^{-(\theta+\lambda+\mu)(t-y)} \left[I_{l-1}\big(\alpha(t-y)\big) - I_{l+1}\big(\alpha(t-y)\big)\right] dy$$
$$+\delta \int_0^t \sum_{k=2}^{\infty} \left[\{P_{k,1}(y) + P_{k,2}(y)\}\gamma^{l-k}\left[I_{l-k}\big(\alpha(t-y)\big) - I_{l+k}\big(\alpha(t-y)\big)\right]\right] e^{-(\theta+\lambda+\mu)(t-y)} dy, \quad l \geq 2 \quad (20)$$

From (9), (11) (13), $\bar{P}_{0,1}(s) = \frac{\mu q}{s+\theta+\delta+\lambda\beta}\, \bar{P}_{1,a}(s)$ (21)

$$\bar{P}_{0,2}(s) = \left[\frac{\mu p}{s+\theta+\lambda\beta} + \frac{\mu q \delta}{(s+\theta+\lambda\beta)(s+\theta+\delta+\lambda\beta)}\right] \bar{P}_{1,a}(s) \quad (22)$$

$$\bar{P}_{1,a}(s) = \frac{\mu}{s+\theta+\mu+\lambda}\, \bar{P}_{2,a}(s) + \bar{A}(s)\bar{P}_{1,a}(s) + \frac{\xi}{s+\theta+\mu+\lambda}\, \bar{Q}(s) \quad (23)$$

where

$$\bar{A}(s) = \frac{\delta\lambda\beta}{s+\theta+\delta+\lambda\beta}\left[\frac{\mu q}{s+\theta+\delta+\lambda\beta} + \frac{\mu p}{s+\theta+\lambda\beta} + \frac{\mu q \delta}{(s+\theta+\lambda\beta)(s+\theta+\delta+\lambda\beta)}\right] \quad (24)$$

Putting $l = 2$ in (20), taking Laplace transform and after some algebra, we get

$$\bar{P}_{2,a}(s) = \left[\lambda \bar{P}_{1,a}(s)\bar{D}_{2,1}(s) + \delta \sum_{k=2}^{\infty}\{\bar{P}_{k,1}(y) + \bar{P}_{k,2}(y)\}\gamma^{2-k}\bar{D}_{2,k}(s)\right]\bar{E}(s) \quad (25)$$

where

$$\bar{E}(s) = \sum_{m=0}^{\infty}(-1)^m \left(\mu\gamma\bar{D}_{2,2}(s)\right)^m, \quad (26)$$

$$\bar{D}_{l,k}(s) = \frac{\left(w-\sqrt{w^2-\alpha^2}\right)^{l-k}}{\alpha^{l-k}\sqrt{w^2-\alpha^2}} - \frac{\left(w-\sqrt{w^2-\alpha^2}\right)^{l+k}}{\alpha^{l+k}\sqrt{w^2-\alpha^2}}, w = s+\theta+\lambda+\mu \quad (27)$$

From (23) and (25), after solving, we get $\bar{P}_{1,a}(s) = \frac{\xi}{s+\theta+\mu+\lambda}\, \bar{Q}(s) \sum_{l=0}^{\infty}[\bar{F}(s)]^l$ (28)

where $\bar{F}(s) = \bar{M}(s) + \frac{\mu^2\delta}{(s+\theta+\lambda\beta)(s+\theta+\mu+\lambda)}\bar{E}(s)\left(\frac{\lambda\beta}{s+\theta+\delta+\lambda\beta}\right)^l \gamma^{2-k}\bar{D}_{2,k}(s)$,

$\bar{M}(s) = \frac{\lambda\mu}{(s+\theta+\mu+\lambda)}\bar{D}_{2,1}(s)\bar{E}(s) + \bar{A}(s)$. Inverting (28), we get

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$$P_{1,a}(t) = \xi e^{-(\theta+\mu+\lambda)t} * Q(t) * \sum_{l=0}^{\infty} [F(t)]^{*l} \tag{29}$$

$F(t) = M(t) + \frac{\mu^2\delta}{\mu+\lambda-\lambda\beta}[e^{-(\theta+\lambda\beta)t} - e^{-(\theta+\mu+\lambda)t}] * E(t) * \gamma^{2-k}(\lambda\beta)^k \frac{t^{k-1}}{(k-1)!} e^{-(\theta+\delta+\lambda\beta)t} * D_{2,k}(t)$ with

$M(t) = \lambda\mu e^{-(\theta+\mu+\lambda)t} * D_{2,1}(t) * E(t) + A(t)$

$E(t) = \sum_{m=0}^{\infty}(-1)^m \left(\mu\gamma D_{2,2}(t)\right)^{*m}$, $A(t) = \mu\lambda\beta[e^{-(\theta+\lambda\beta)t} - e^{-(\theta+\delta+\lambda\beta)t}]$

$D_{l,k}(t) = [I_{l-k}(\alpha t) - I_{l+k}(\alpha t)]e^{-(\theta+\lambda+\mu)t}$, where $'*'$ and $'*l'$ denote Laplace convolution and l fold Laplace convolution respectively. Again, inverting (21) and (22), we get

$$P_{0,1}(t) = \mu q e^{-(\theta+\delta+\lambda\beta)t} * P_{1,a}(t) \tag{30}$$

$$P_{0,2}(t) = \left[\mu e^{-(\delta+\lambda\beta)t} - \mu q e^{-(\theta+\delta+\lambda\beta)t}\right] * P_{1,a}(t) \tag{31}$$

Using (30) and (31) in (17) and (16) respectively we get $P_{l,1}(t)$ and $P_{l,2}(t), l \geq 1$ explicitly in terms of $P_{1,a}(t)$. Now it remains to find $P_{2,a}(t)$. Using (14), (15), (21) and (22) in (31), we obtain after some manupulations

$$\bar{P}_{2,a}(s) = \left[\lambda\bar{D}_{2,1}(s) + \bar{E}(s)\bar{A}(s)\sum_{k=2}^{\infty}\left(\frac{\lambda\beta}{s+\theta+\delta+\lambda\beta}\right)^{k-1}\gamma^{2-k}\bar{D}_{2,k}(s)\right]\bar{P}_{1,a}(s) \tag{32}$$

Inverting (32),

$$P_{2,a}(t) = \left[\lambda D_{2,1}(t) + E(t) * A(t) * \sum_{k=2}^{\infty}\gamma^{2-k}(\lambda\beta)^{k-1}t^{k-2}e^{-(\theta+\delta+\lambda\beta)t} * D_{2,k}(t)\right] * P_{1,a}(t) \tag{33}$$

Hence all the probabilities $P_{l,i}(t), l \geq 0, i = 1, 2, a, (l, a) \neq (0, a)$ have been expressed in terms of $P_{1,a}(t)$ explicitly. Equation (29) gives $P_{1,a}(t)$ explicitly in terms of $Q(t)$ which is independently given by (8).

IV. PERFORMANCE MEASURES

A. Time-dependent Mean

$\mathbb{A}(t)$ Average number of clients $\mathcal{M}(t)$ in the system at any time t is given by $\mathbb{A}(t) = \sum_{l=1}^{\infty} l(P_{l,1}(t) + P_{l,2}(t) + P_{l,a}(t))$.

Differentiating this with respect to t, we have

$$\frac{d\mathbb{A}(t)}{dt} = \sum_{l=1}^{\infty} l(P'_{l,1}(t) + P'_{l,2}(t) + P'_{l,a}(t)) \quad (34)$$

Using equations (2), (4), (6) and making some simplifications, then integrating between limits 0 to t, we get

$$\mathbb{A}(t) = -\lambda \int_0^t P_{l,a}(y)\, e^{-\theta(t-y)} dy - (\lambda - \mu) \int_0^t \sum_{l=1}^{\infty} P_{l,a}(y)\, e^{-\theta(t-y)} dy - \lambda\beta \int_0^t \sum_{l=0}^{\infty} \left[P_{l,1}(y) + P_{l,2}(y)\right] e^{-\theta(t-y)} dy - \delta \int_0^t Q(y)\, e^{-\theta(t-y)} dy \quad (35)$$

B. Time-dependent Variance

Time-dependent variance $\mathrm{V}(t)$ is given by

$$\mathrm{V}(t) = \mathrm{K}(t) - [\mathbb{A}(t)]^2 \quad (36)$$

$$\text{with } \mathrm{K}(t) = \sum_{l=1}^{\infty} l^2 (P_{l,1}(t) + P_{l,2}(t) + P_{l,a}(t)) \quad (37)$$

Differentiating (37) with respect to t, we have

$$\frac{d\mathrm{K}(t)}{dt} = \sum_{l=1}^{\infty} l^2 (P'_{l,1}(t) + P'_{l,2}(t) + P'_{l,a}(t)) \quad (38)$$

Using (2), (4), (6) and making some manipulations, then integrating between limits 0 to t, we get

$$\mathrm{K}(t) = -\lambda \int_0^t P_{l,a}(y)\, e^{-\theta(t-y)} dy - \mu \int_0^t \sum_{l=1}^{\infty} (2l-1) P_{1,a}(y)\, e^{-\theta(t-y)} dy + \lambda \int_0^t \sum_{l=0}^{\infty} (2l+1)\, P_{l,a}(y) e^{-\theta(t-y)} dy + \lambda\beta \int_0^t \sum_{l=1}^{\infty} (2l+1) \left[P_{l,1}(y) + P_{l,2}(y)\right] e^{-\theta(t-y)} dy + \xi \int_0^t Q(y)\, e^{-\theta(t-y)} dy \quad (39)$$

Equations (35), (36) and (39) together give time-dependent variance.

V. CONCLUSION AND FUTURE SCOPE

By applying the methods of PGF (probability generating function), iterative procedure and Laplace transform we have made transient analysis of the model and obtained system probabilities in transient scenario. Time-dependent mean and variance have been obtained in explicit terms. As a future work one can extend the model by incorporating reneging with (or without) balking in time-dependent or time-independent scenario

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