UNIT – I

Fourier Series: Introduction to Fourier series, Fourier series for Discontinuous functions, Fourier series for even and odd function, Half range series

Fourier Transform: Definition and properties of Fourier transform, Sine and Cosine transform

CHAPTER 1

Fourier Series

Introduction: In many physical and engineering problems, especially in the study of periodic function, it is necessary to express a real valued function in series of sines and cosines, which can be expressed in the form.

$$\frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + ... + b_1 \sin x + b_2 \sin 2x + ...$$

where $a_0, a_1, a_2, ...$ and $b_1, b_2, ...$ are some constants.

Periodic functions: A function $f(x)$ is said to be periodic if $f(x+l) = f(x), \forall x$ being a real number. If l is the least positive number called the period of the function $f(x)$.

For example: $sinx, \cos x, \sec x, \cos ecx$ is a periodic function with period 2π.

Also, $\sin nx, \cos nx$ are periodic function with period $\frac{2\pi}{n}$.

Trigonometric Series: A series of the form

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is called a trigonometric series.

Some useful Definite Integrals: Let m and n be any two integers and the interval $(c, c+2\pi)$. Then the following definite integrals are as follows:

(i) $\int_{\alpha}^{\alpha+2\pi} \sin nx\,dx = \left[-\frac{\cos nx}{n}\right]_{\alpha}^{\alpha+2\pi} = 0, \quad n \neq 0$

(ii) $\int_{\alpha}^{\alpha+2\pi} \cos nx\,dx = 0, \quad n \neq 0$

(iii) $\int_{\alpha}^{\alpha+2\pi} \sin mx \sin nx\,dx = \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right]_{\alpha}^{\alpha+2\pi} = 0, \ m \neq n$

(iv) $\int_{\alpha}^{\alpha+2\pi} \cos mx \cos nx\,dx = \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} + \frac{\sin(m+n)x}{m+n}\right]_{\alpha}^{\alpha+2\pi} = 0, \ m \neq n$

(v) $\int_{\alpha}^{\alpha+2\pi} \sin^2 nx\,dx = \frac{1}{2}\int_{\alpha}^{\alpha+2\pi} (1-\cos 2nx)\,dx = \pi, \ n \neq 0$

(vi) $\int_{\alpha}^{\alpha+2\pi} \cos^2 nx\,dx = \frac{1}{2}\int_{\alpha}^{\alpha+2\pi} (1+\cos 2nx)\,dx = \pi, \ n \neq 0$

(vii) $\int_{\alpha}^{\alpha+2\pi} \sin mx \cos nx\,dx = \frac{1}{2}\left[\frac{\cos(m+n)x}{m+n} + \frac{\cos(m-n)x}{m-n}\right]_{\alpha}^{\alpha+2\pi} = 0, \ m \neq n$

(viii) $\int_{\alpha}^{\alpha+2\pi} \sin nx \cos nx\,dx = \left[\frac{\sin^2 nx}{2n}\right]_{\alpha}^{\alpha+2\pi} = 0, \ n \neq 0$

(ix) $\int e^{ax} \sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$

(x) $\int e^{ax} \cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

(xi) $\sin n\pi = 0$ and $\cos n\pi = (-1)^n, \quad n \in N.$

Fourier Series: The Fourier Series for the function $f(x)$ in the interval $(\alpha, \alpha+2\pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

where

$$a_0 = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx\, dx$$

and $$b_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx\, dx$$

The values of a_0, a_n, b_n are known as Euler's Coefficients or Fourier Coefficients.

Proof: Let $f(x)$ be represented in the interval $(\alpha, \alpha+2\pi)$ by the Fourier Series:

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \qquad \text{.....(i)}$$

To find the coefficients a_0, a_n, b_n. We assume that equation (i) can be integrated term by term in the given interval $(\alpha, \alpha+2\pi)$.

To find a_0: integrated both sides of (i), we get

$$\int_{\alpha}^{\alpha+2\pi} f(x)\,dx = \frac{a_0}{2}\int_{\alpha}^{\alpha+2\pi} dx + \int_{\alpha}^{\alpha+2\pi}\left(\sum_{n=1}^{\infty} a_n \cos nx\right)dx + \int_{\alpha}^{\alpha+2\pi}\left(\sum_{n=1}^{\infty} b_n \sin nx\right)dx$$

$$= \frac{a_0}{2}(\alpha + 2\pi - \alpha) + 0 + 0$$

$$= a_0.\pi$$

Hence $$a_0 = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\,dx$$

To find a_n : multiply both sides of (i) by $\cos nx$ and integrating.

$$\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx\,dx = \frac{a_0}{2}\int_{\alpha}^{\alpha+2\pi}\cos nx\,dx +$$

$$\int_{\alpha}^{\alpha+2\pi}\left(\sum_{n=1}^{\infty} a_n \cos nx\right)\cos nx\,dx + \int_{\alpha}^{\alpha+2\pi}\left(\sum_{n=1}^{\infty} b_n \sin nx\right)\cos nx\,dx$$

$$= 0 + \pi a_n + 0$$

Hence $a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx\,dx$.

To find b_n : multiply both sides of (i) by $\sin nx$ and integrating

$$\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx\,dx = \frac{a_0}{2}\int_{\alpha}^{\alpha+2\pi} \sin nx\,dx +$$

$$\int_{\alpha}^{\alpha+2\pi}\left(\sum_{n=1}^{\infty} a_n \cos nx\right)\sin nx\,dx + \int_{\alpha}^{\alpha+2\pi}\left(\sum_{n=1}^{\infty} b_n \sin nx\right)\sin nx\,dx$$

$$= 0 + 0 + \pi b_n$$

Hence $b_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx\,dx$.

Deductions: (i) Taking $\alpha = 0$, then interval becomes $0 \le x \le 2\pi$ and (i) becomes

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx,\ a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx,\ b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

(ii) taking $\alpha = -\pi$, then interval becomes $-\pi \le x \le \pi$ and (i) becomes

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx,\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx,\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

Dirichlet's conditions: (RGPV Feb 2007)

Any function $f(x)$ can be expressed as a Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

in the interval $(0, 2\pi)$ or $(-\pi, \pi)$, where a_0, a_n, b_n are constants provided that $f(x)$ satisfies the following conditions:

(i) $f(x)$ is periodic.

(ii) $f(x)$ and its integrals are finite and single valued.

(iii) $f(x)$ has a finite number of discontinuities.

(iv) $f(x)$ has a finite number of maxima and minima.

when these conditions are satisfied, the Fourier series converges to $f(x)$ at every point of continuity. At a point of discontinuity, the sum of the series is equal to the mean of the limits on the right and left.

i.e. $\frac{1}{2}\left[f(x+0) + f(x-0)\right]$

where $f(x+0)$ and $f(x-0)$ be the right and left limit.

Parseval's theorem:- Let the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx,$$

If $f(x)$ converges uniformly to $f(x)$ at every point of the interval $(0, 2\pi)$ then

$$\frac{1}{\pi}\int_0^{2\pi} \left[f(x)\right]^2 dx = \frac{a_0^{\ 2}}{2} + \sum_{n=1}^{\infty}\left(a_n^{\ 2} + b_n^{\ 2}\right).$$

Illustrative Examples

Example 1. Find a Fourier Series to represent $x - x^2$ from $x = -\pi$ to $x = \pi$. Hence show

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + ... = \frac{\pi^2}{12}$$ (RGPV Feb 2005, June 2006, Feb 2010)

Sol: Let $f(x) = x - x^2, \quad -\pi < x < \pi$.

Fourier Series over the interval $(-\pi, \pi)$ be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad .....(i)$$

Now,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(x - x^2\right)dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{-2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(x - x^2\right)\cos nx\, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\cos nx\, dx - \int_{-\pi}^{\pi} x^2 \cos nx\, dx\right]$$

$$= -\frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\, dx,$$ (since $x\cos nx$ is an odd function.)

$$= -\frac{1}{\pi}\left[\left\{x^2\frac{\sin nx}{n}\right\}_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2x\frac{\sin nx}{n}dx\right]$$

$$= -\frac{1}{\pi}\left[x^2\frac{\sin nx}{n} + 2x\frac{\cos nx}{n^2} + 2\frac{\sin nx}{n^3}\right]_{-\pi}^{\pi}$$

$$= -\frac{1}{\pi n^2}\left[4\pi\cos n\pi\right] \qquad \left[\because \sin n\pi = 0,\ \cos n\pi = (-1)^n\right]$$

$$= -\frac{4}{n^2}(-1)^n$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(x - x^2\right)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\sin nx - \int_{-\pi}^{\pi} x^2\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx \qquad \left[\text{since } x^2\sin nx \text{ is an odd function.}\right]$$

$$= \frac{1}{\pi}\left[\left\{-x\frac{\cos nx}{n}\right\}_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{\cos nx}{n^2}dx\right]$$

$$= \frac{1}{\pi}\left[-x\frac{\cos nx}{n} + \frac{\sin nx}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-\pi\frac{\cos n\pi}{n} - \pi\frac{\cos n\pi}{n}\right]$$

$$= -\frac{2}{n}(-1)^n$$

Hence (i) becomes

$$x - x^2 = -\frac{\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$$

$$= -\frac{\pi^2}{3} - 4\left[-\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \ldots\right] - 2\left[-\frac{\sin x}{1} + \frac{\sin 2x}{2} - \frac{\sin 3x}{3} + \ldots\right]$$

$$= -\frac{\pi^2}{3} + 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \ldots\right] + 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \ldots\right]$$

Putting $x = 0$, we get

$$0 = -\frac{\pi^2}{3} + 4\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - ...\right)$$

$$\Rightarrow \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - ... = \frac{\pi^2}{12}$$

Example 2. Find a Fourier Series to represent the function $f(x) = x + x^2$ in the interval $-\pi < x < \pi$.

Hence show that $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ...$ (RGPV Feb 2006)

Sol: Let $f(x) = x + x^2, \ -\pi < x < \pi$.

Fourier Series over the interval $(-\pi, \pi)$ be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad .....(i)$$

Now,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(x + x^2\right)dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2} + \frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(x + x^2\right)\cos nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\left(x + x^2\right)\cos nx\, dx \qquad \left[\text{Since } x\cos nx \text{ is an odd function.}\right]$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x^2 \cos nx\, dx$$

$$= \frac{2}{\pi}\left[x^2\frac{\sin nx}{n} - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[2\pi\frac{\cos n\pi}{n^2}\right] = \frac{4}{n^2}(-1)^n$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(x+x^2\right)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx \qquad \left[\text{since } x^2 \sin nx \text{ is an odd function.}\right]$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x\sin nx\,dx$$

$$= \frac{2}{\pi}\left[-x\frac{\cos nx}{n} - \left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\pi\cos n\pi}{n}\right] = -\frac{2}{n}(-1)^n$$

Hence (i) becomes

$$x + x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$$

$$= \frac{\pi^2}{3} + 4\left[-\cos x + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \ldots\right] - 2\left[-\sin x + \frac{\sin 2x}{2} - \frac{\sin 3x}{3} + \ldots\right]$$

$$= \frac{\pi^2}{3} - 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \ldots\right] + 2\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \ldots\right] \qquad \ldots(2)$$

Which is the required Fourier Series.

Now putting $x = \pi$ and $x = -\pi$ in equation (2), we have

$$\pi + \pi^2 = \frac{\pi^2}{3} + 4\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right] \qquad \ldots..(3)$$

and $$-\pi + \pi^2 = \frac{\pi^2}{3} + 4\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right] \qquad \ldots..(4)$$

Adding (3) and (4), we have

$$2\pi^2 = \frac{2\pi^2}{3} + 8\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right]$$

$$\Rightarrow \frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$$

Example 3 Find Fourier Series represented of $f(x) = x\sin x,\ 0 < x < 2\pi$.

(RGPV June 2004, June 2007, Dec 2008)

Sol: Let $f(x) = x\sin x,\ 0 < x < 2\pi$

The Fourier series of $f(x)$ in $(0, 2\pi)$ be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \text{.....(1)}$$

Now

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin x\,dx = \frac{1}{\pi}\left[x(-\cos x) - (-\sin x)\right]_0^{2\pi} = -2$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin x\,\cos nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(2\sin x\cos nx)\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\sin(n+1)x - \sin(n-1)x\right]dx$$

$$\left[\because 2\cos A\, sinB = \sin(A+B) - \sin(A-B)\right]$$

$$= \frac{1}{2\pi}\left[x\left\{-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\} - \left\{-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi\left\{-\frac{\cos 2(n+1)\pi}{n+1} + \frac{\cos 2(n-1)\pi}{n-1}\right\}\right]$$

$$= -\frac{1}{n+1} + \frac{1}{n-1} = \frac{2}{n^2-1}, \qquad \text{if } n \neq 1$$

But if $n=1$, then

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos x\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\sin 2x\,dx$$

$$= \frac{1}{2\pi}\left[x\left(-\frac{\cos 2x}{2}\right)-\left(-\frac{\sin 2x}{4}\right)\right]_0^{2\pi} = -\frac{1}{2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin x\sin nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\cos(n-1)x-\cos(n+1)x\right]dx$$

$$= \frac{1}{2\pi}\left[x\left\{\frac{\sin(n-1)x}{n-1}-\frac{\sin(n+1)x}{n+1}\right\}-\left\{-\frac{\cos(n-1)x}{(n-1)^2}+\frac{\cos(n+1)x}{(n+1)^2}\right\}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\cos 2(n-1)\pi}{(n-1)^2}-\frac{\cos 2(n+1)\pi}{(n+1)^2}-\frac{1}{(n-1)^2}+\frac{1}{(n+1)^2}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{(n-1)^2}-\frac{1}{(n+1)^2}-\frac{1}{(n-1)^2}+\frac{1}{(n+1)^2}\right] = 0, \qquad \text{if } n\neq 1$$

But if $n=1$, then

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin x\sin x\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x2\sin^2 x\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(1-\cos 2x)\,dx$$

$$= \frac{1}{2\pi}\left[x\left(x-\frac{\sin 2x}{2}\right)-\left(\frac{x^2}{2}+\frac{\cos 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi.2\pi-\frac{4\pi^2}{2}-\frac{1}{4}+\frac{1}{4}\right] = \pi$$

Hence Fourier Series (1) becomes

$$f(x) = \frac{a_0}{2} + a_1 \cos x + b_1 \sin x + \sum_{n=2}^{\infty} a_n \cos nx + 0 \qquad (\because b_n = 0)$$

$$x \sin x = -1 - \frac{1}{2}\cos x + \pi \sin x + \sum_{n=2}^{\infty} \frac{2}{(n^2-1)} \cos nx$$

$$= -1 - \frac{1}{2}\cos x + \pi \sin x + \frac{2}{(2^2-1)}\cos 2x + \frac{2}{(3^2-1)}\cos 3x + ...$$

Example 4. Find the Fourier Series to represent the function, if

$$f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$

Hence deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ...$ (RGPV Dec 2004, Feb 2007, Dec 2008)

Sol: Let $f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ …..(1)

where

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx\right]$$

$$= \frac{1}{2\pi}\int_{-\pi}^{0} (-\pi)\,dx + \frac{1}{2\pi}\int_{0}^{\pi} x\,dx$$

$$= -\frac{1}{2}[x]_{-\pi}^{0} + \frac{1}{2\pi}\left[\frac{x^2}{2}\right]_0^{\pi}$$

$$= -\frac{\pi}{2} + \frac{\pi^2}{4\pi} = -\frac{\pi}{4}$$

$$a_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi \cos nx)\,dx + \int_{0}^{\pi} x \cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left\{-\frac{\pi \sin nx}{n}\right\}_{-\pi}^{0} + \left\{x\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right\}_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[0 + \frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right]$$

$$= \frac{\cos n\pi - 1}{\pi n^2}$$

If n is even $\cos n\pi = 1$ then $a_n = 0$ *i.e.* $a_2 = a_4 = ... = 0$

If n is odd $\cos n\pi = -1$ then $a_n = -\frac{2}{\pi n^2}$ *i.e.* $a_1 = -\frac{2}{\pi.1^2}$, $a_3 = -\frac{2}{\pi.3^2}$ etc.

and

$$b_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi \sin nx)\, dx + \int_{0}^{\pi} x \sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\left\{\frac{\pi \cos nx}{n}\right\}_{-\pi}^{0} + \left\{x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right\}_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n} - \frac{\pi \cos n\pi}{n} - \frac{\pi \cos n\pi}{n} + 0\right] = \frac{1 - 2\cos n\pi}{n}$$

$\therefore$ if n is even $\cos n\pi = 1$ then $b_n = -\frac{1}{n}$, *i.e.* $b_2 = -\frac{1}{2}$, $b_4 = -\frac{1}{4}$ etc.

If n is odd $\cos n\pi = -1$ then $b_n = \frac{3}{n}$, *i.e.* $b_1 = \frac{3}{1}$, $b_3 = \frac{3}{3}$, $b_5 = \frac{3}{5}$ etc.

Now putting values in (1) then required Fourier Series is

$$f(x) = -\frac{\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + ...\right] +$$
$$\left[\frac{3}{1}\sin x - \frac{1}{2}\sin 2x + \frac{3}{3}\sin 3x - ...\right] \qquad .....(2)$$

To find the sum: At $x = 0$ the series converges to

$$\frac{f(0+0) + f(0-0)}{2} = \frac{-\pi + 0}{2} = -\frac{\pi}{2}$$

and at $x = \pi$ the series converges to

$$\frac{f(-\pi+0)+f(\pi-0)}{2} = \frac{-\pi+\pi}{2} = 0$$

The L.H.S. of (2) is $\frac{-\pi}{2}$ at $x = 0$ and is 0 at $x = \pi$

Therefore putting $x = 0$ in R.H.S. of (2) we get

$$-\frac{\pi}{2} = -\frac{\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right]$$

or

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

Example 5 Obtain the Fourier Series for $f(x) = e^x$ in the interval $-\pi < x < \pi$.

Sol: Suppose that

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \text{.....(1)}$$

Now

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x dx$$

$$= \frac{1}{2\pi}\left[e^x\right]_{-\pi}^{\pi} = \frac{1}{2\pi}\left(e^{\pi} - e^{-\pi}\right) = \frac{\sinh \pi}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^x}{1+n^2}(1.\cos nx + n \sin nx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(1+n^2)}\left[e^{\pi}\cos n\pi - e^{-\pi}\cos(-n\pi)\right]$$

$$= \frac{\cos n\pi}{\pi(1+n^2)}\left(e^{\pi} - e^{-\pi}\right) = \frac{2(-1)^n \sinh \pi}{\pi(1+n^2)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin nx\ dx$$

$$= \frac{1}{\pi}\left[\frac{e^x}{1+n^2}(1.\sin nx - n\cos nx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(1+n^2)}\left[(-n)e^{\pi}\cos n\pi - (-n)e^{-\pi}\cos(-n\pi)\right]$$

$$= \frac{(-n)\cos n\pi}{\pi(1+n^2)}\left(e^{\pi} - e^{-\pi}\right)$$

$$= \frac{-2(-1)^n n\sinh\pi}{\pi(1+n^2)}$$

Now putting values of $a_0, a_n \& b_n$ in (1) we get

$$f(x) = e^x = \frac{\sinh\pi}{\pi} + \sum_{n=1}^{\infty}\frac{2(-1)^n \sinh\pi}{\pi(1+n^2)}(\cos nx - n\sin nx)$$

$$e^x = \frac{2\sinh x}{\pi}\left[\frac{1}{2} - \frac{1}{2}(\cos x - \sin x) + \frac{1}{5}(\cos 2x - 2\sin 2x) + \ldots\right]$$

Which is the required Fourier Expansion.

Example 6. Find a series of $\sin es$ and $\cos ines$ of multiples of x which will represent $f(x)$ in the interval $(-\pi, \pi)$ then

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ \frac{1}{4}\pi x & 0 < x < \pi \end{cases}$$

Sol: Let $f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ …..(1)

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{0} f(x)\,dx + \frac{1}{2\pi}\int_{0}^{\pi} f(x)\,dx$$

$$= 0 + \frac{1}{2\pi}\int_{0}^{\pi} \frac{1}{4}\pi x\,dx$$

$$= \frac{1}{8\pi}.\pi.\frac{\pi^2}{2} = \frac{\pi^2}{16} \qquad \text{.....(2)}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

$$= 0 + \frac{1}{\pi}\int_{0}^{\pi} \frac{1}{4}\pi x\cos nx\,dx$$

$$= \frac{1}{4}\left[x.\frac{\sin nx}{n} - 1.\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{1}{4}\left[0 + \frac{1}{n^2}(\cos n\pi - 1)\right]$$

$$= \frac{1}{4n^2}\left[(-1)^n - 1\right]$$

Now if n is odd

$$(-1)^n = -1$$

$$a_n = \frac{-1-1}{4n^2} = -\frac{1}{2n^2}$$

$$a_1 = -\frac{1}{2},\ \ a_2 = -\frac{1}{2.3^2},....$$

And if n is even

$$(-1)^n = 1$$

$$a_n = \frac{1-1}{4n^2} = 0, \quad \therefore a_2 = a_4 = ... = 0$$

Similarly we have

$$b_n = -\frac{\pi}{4n}(-1)^n$$

$$b_1 = \frac{\pi}{4}, b_2 = -\frac{\pi}{4.2},\ b_3 = \frac{\pi}{4.3},\ b_4 = -\frac{\pi}{4.4}......$$

Now putting values in (1) we get

$$f(x) = a_0 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + ...$$

$$= \frac{\pi^2}{16} + \left(-\frac{1}{2}\cos x + \frac{\pi}{4}\sin x\right) + \left(0 - \frac{\pi}{4.2}\sin 2x\right) + .$$

$$\left(-\frac{1}{2.3^2}\cos 3x + \frac{\pi}{4.3}\sin 3x\right) + ...$$

Example 7. Find the Fourier Series expression of the function $f(x)$ given by

$f(x) = |x|$ for $-\pi \le x \le \pi$ (RGPV Dec 2005, Dec 2007)

Sol: Let the Fourier Series be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad .....(1)$$

Now,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-x)\,dx + \int_{0}^{\pi} x\,dx\right]$$

$$= \frac{1}{\pi}\left[\left(-\frac{x^2}{2}\right)_{-\pi}^{0} + \left(\frac{x^2}{2}\right)_{0}^{\pi}\right] = \frac{1}{\pi}\left[\frac{\pi^2}{2} + \frac{\pi^2}{2}\right] = \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-x)\cos nx\,dx + \int_{0}^{\pi} x\cos nx\,dx\right]$$

$$= \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$$= \frac{2}{\pi}\left[x\frac{\sin nx}{n} - \left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi} = \frac{2}{\pi n^2}\left[\cos n\pi - 1\right] = \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-x)\sin nx\,dx + \int_0^{\pi} x\sin nx\,dx\right]$$

$$= 0$$

Putting the value of a_0, a_n, b_n in (1), we get

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{\pi n^2}\left[(-1)^n - 1\right]\cos nx$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + ...\right]$$

Even and Odd functions: A function $f(x)$ is said to be an even function if $f(-x) = f(x)$, *e.g.* $\cos x,\ x^2,\ x^3\sin x$, etc.

A function $f(x)$ is said to be an odd function if

$f(-x) = -f(x)$, *e.g.* $\sin x,\ x^3,\ x^3\cos x,\ x^2\sin x$ etc.

The Fourier Series of $f(x)$ in the interval $(-\pi, \pi)$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx \qquad .....(1)$$

For even function:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

and $b_n = 0$

$\therefore$ from (1), The Fourier Series becomes

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

Hence if $f(x)$ is an even function then Fourier Series contains only $\cos ine$ term.

For Odd function:

$$a_0 = 0, \quad a_n = 0 \text{ and } b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$

$\therefore$ from (1), the Fourier series becomes

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

Hence if $f(x)$ is an odd function then Fourier series contains only $Sine$ term.

Half Range Fourier Series: If a half range series for a function $f(x)$ is desired, then the function is defined in the interval $(0, l)$, (*i.e.* half of the interval $(-l, l)$ and is said to be half range.)

Thus we can obtain the Fourier series either $\cos ine$ series or $\sin e$ series only.

Cosine Series: If $f(x)$ is an even function defined on interval $-l \le x \le l$. Then $\cos ine$ Fourier series in half range interval $(0, l)$ becomes

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

where $a_0 = \frac{2}{l}\int_0^{l} f(x)\, dx$ and $a_n = \frac{2}{l}\int_0^{l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx, \; n = 1, 2, 3, \ldots$

Sine Series: If $f(x)$ is an odd function defined on interval $-l \le x \le l$, then $Sine$ Fourier series in half range interval $(0, l)$ becomes

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where $$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx, \qquad n = 1,2,3,......$$

Example 8. Find Fourier Series representation of $f(x) = |x|$ in the interval $-l \le x \le l$.

Or

Find $\cos ine$ Fourier series of $f(x) = |x|$ in $(0,l)$. (RGPV 2001)

Sol: Let $f(x) = |x|$, which is an even function in the interval $(-l,l)$

$\therefore \cos ine$ Fourier series of $f(x)$ in half range interval $(0,l)$ becomes

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \qquad .....(1)$$

Now,

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{l}\int_0^l |x|\,dx$$

$$= \frac{2}{l}\int_0^l x\,dx \qquad (\because |x| = x,\ 0 \le x \le l)$$

$$= \frac{2}{l}\cdot\frac{l^2}{2} = l$$

and

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\int_0^l x\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\left[\left\{x\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right\}_0^l - \int_0^l \frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\,dx\right]$$

$$= \frac{2}{l}\left[0 + \frac{\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right]_0^l$$

$$= \frac{2}{l} \cdot \frac{l^2}{n^2\pi^2}(\cos n\pi - \cos 0)$$

$$= \frac{2l}{n^2\pi^2}\left[(-1)^n - 1\right]$$

when n is odd, $a_n = -\frac{4l}{n^2\pi^2}$

when n is even, $a_n = 0$

Hence Fourier series (1) becomes

$$f(x) = \frac{l}{2} - \frac{4l}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\left(\frac{n\pi x}{l}\right)$$

$$= \frac{l}{2} - \frac{4l}{\pi^2}\left[\frac{\cos\frac{\pi x}{l}}{1^2} + \frac{\cos\frac{3\pi x}{l}}{3^2} + \frac{\cos\frac{5\pi x}{l}}{5^2} + \ldots\ldots\right]$$

Example 9. A periodic function of period 4 is defined as $f(x) = |x|,\ -2 < x < 2.$ Find its Fourier Series expansion. (RGPV Dec 2002)

Sol: Taking $l = 2$ in the above example 8 and proceed.

Example 10. Find a Fourier series to represent $f(x) = x^2$ in the interval $-l < x < l.$

(RGPV June 2005, Feb 2010)

Sol: Let $f(x) = x^2$ is an even function in the interval $(-l, l)$.

$$\therefore\ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \qquad \ldots\ldots(1)$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{l}\int_0^l x^2\,dx$$

Now,

$$= \frac{2}{l}\left(\frac{x^3}{3}\right)_0^l = \frac{2l^2}{3}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^l x^2\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[x^2 \frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}} - 2x\left(-\frac{\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}} \right) + 2\left(-\frac{\sin\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}} \right) \right]_0^l$$

$$= \frac{4l^2}{n^2\pi^2}(-1)^n$$

Hence the Fourier series (1) becomes

$$x^2 = \frac{l^3}{3} + \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi x}{l}$$

$$= \frac{l^3}{3} + \frac{4l^2}{\pi^2}\left[-\frac{1}{1^2}\cos\frac{\pi x}{l} + \frac{1}{2^2}\cos\frac{2\pi x}{l} - \frac{1}{3^2}\cos\frac{3\pi x}{l} + \dots\dots \right]$$

$$x^2 = \frac{l^3}{3} - \frac{4l^2}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{l} - \frac{1}{2^2}\cos\frac{2\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} - \dots\dots \right]$$

Example 11. Obtain the half range sine series for $f(x) = \pi x - x^2$ in the interval $0 < x < \pi.$ (RGPV June 2005)

Sol^n:- Let the Fourier *Sine* series be

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi}(\pi x - x^2)\sin nx\, dx$$

$$= \frac{2}{\pi}\left[\left\{ (\pi x - x^2)\left(-\frac{\cos nx}{n} \right) \right\}_0^{\pi} - \int_0^{\pi}(\pi - 2x)\left(-\frac{\cos nx}{n} \right) dx \right]$$

$$= \frac{2}{\pi}\left[(0-0) + \left\{ (\pi - 2x)\frac{\sin nx}{n^2} \right\}_0^{\pi} - \int_0^{\pi}(-2)\frac{\sin nx}{n^2} dx \right]$$

$$= \frac{2}{\pi}\left[0 - 2\frac{\cos nx}{n^3}\right]_0^{\pi}$$

$$= -\frac{4}{\pi n^3}\left[\cos n\pi - 1\right]$$

$$= -\frac{4}{\pi n^3}\left[(-1)^n - 1\right]$$

Hence the required Fourier half range *Sine* series is

$$\pi x - x^2 = \sum_{n=1}^{\infty}\left(-\frac{4}{\pi n^3}\right)\left[(-1)^n - 1\right]\sin nx$$

$$= -\frac{4}{\pi}\left[-\frac{2}{1^3}\sin x - \frac{2}{3^3}\sin 3x - \frac{2}{5^3}\sin 5x - ......\right]$$

$$= \frac{8}{\pi}\left[\frac{1}{1^3}\sin x + \frac{1}{3^3}\sin 3x + \frac{1}{5^3}\sin 5x + ...\right]$$

Example 12. Express $f(x) = x$ as a:

(i) Half range *Sine* series in $0 < x < 2$. (RGPV June 2006)

(ii) Half range $\cos ine$ series in $0 < x < 2$. (RGPV Jan 2007, June 2009)

Sol: (i) The half range Fourier *Sine* series be

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2} \qquad .....(1)$$

where

$$b_n = \frac{2}{2}\int_0^2 f(x)\sin\frac{n\pi x}{2}\,dx$$

$$= \int_0^2 x\sin\frac{n\pi x}{2}\,dx$$

$$= \left[x\left(-\frac{\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - \left(-\frac{\sin\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right]_0^2$$

$$= -\frac{4}{n\pi}(-1)^n$$

Hence the required half range $\sin e$ series is

$$x = \sum_{n=1}^{\infty} \left(-\frac{4}{n\pi}\right)(-1)^n \sin\frac{n\pi x}{2}$$

$$= \frac{4}{\pi}\left[\sin\frac{\pi x}{2} - \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} - ......\right]$$

(ii) The half range Fourier $\cos ine$ series be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2} \qquad .....(2)$$

where

$$a_0 = \frac{2}{2}\int_0^2 f(x)\,dx = \int_0^2 x\,dx = 2$$

$$a_n = \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx$$

$$= \int_0^2 x\cos\frac{n\pi x}{2}\,dx$$

$$= \left[x\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - \left(-\frac{\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right]_0^2$$

$$= \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right]$$

Hence the required half range $\cos ine$ series is

$$x = 1 + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[(-1)^n - 1\right]\cos\frac{n\pi x}{2}$$

$$= 1 + \frac{4}{\pi^2}\left[-\frac{2}{1^2}\cos\frac{\pi x}{2} - \frac{2}{3^2}\cos\frac{3\pi x}{2} - \frac{2}{5^2}\cos\frac{5\pi x}{2} - ......\right]$$

$$x = 1 - \frac{8}{\pi^2}\left[\frac{1}{1^2}\cos\frac{n\pi}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} - \frac{1}{5^2}\cos\frac{5\pi x}{2} + ...\right]$$

Example 13. Find the half range $Sine$ Fourier series for the function $f(x) = x$ in the interval $0 \le x \le \pi.$ (RGPV June 2007)

Sol: The half range *Sine* series be

$$f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

For the interval $(0,\pi)$,

$$f(x)=\sum_{n=1}^{\infty} b_n \sin nx \qquad .....(1)$$

where

$$b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\ dx$$

$$=\frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx$$

$$=\frac{2}{\pi}\left[\left\{x\left(-\frac{\cos nx}{n}\right)\right\}_0^{\pi}-\int_0^{\pi}\left(-\frac{\cos nx}{n}\right)dx\right]$$

$$=\frac{2}{\pi}\left[-\frac{x\cos nx}{n}+\frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$=-\frac{2}{\pi n}(\pi\cos n\pi)=-\frac{2}{n}(-1)^n$$

Hence, the half range *Sine* series be

$$x=\sum_{n=1}^{\infty}\left(-\frac{2}{n}\right)(-1)^n.\sin nx$$

$$=2\left[\sin x-\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x-......\right]$$

Example 14. Find the half range cosine Fourier series of the function:

$$f(t)=\begin{cases}2t, & 0<t<1\\ 2(2-t), & 1<t<2\end{cases}$$

(RGPV June 2003)

Sol: The half range cosine Fourier series in $(0,2)$ is

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos\frac{n\pi t}{2} \qquad .....(1)$$

where

$$a_0 = \frac{2}{2}\int_0^2 f(t)\,dt \;= \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt$$

$$= \int_0^1 2t\,dt + \int_1^2 2(2-t)\,dt$$

$$= \left[t^2\right]_0^1 + \left[2\left(2t - \frac{t^2}{2}\right)\right]_1^2$$

$$= 1 + 2\left[4 - 2 - 2 + \frac{1}{2}\right] \;=\; 2$$

$$a_n = \frac{2}{2}\int_0^2 f(t)\cos\frac{n\pi t}{2}dt$$

$$= \int_0^1 f(t)\cos\frac{n\pi t}{2}dt + \int_1^2 f(t)\cos\frac{n\pi t}{2}dt$$

$$= \int_0^1 2t\cos\frac{n\pi t}{2}dt + \int_1^2 2(2-t)\cos\frac{n\pi t}{2}dt$$

$$= 2\left[t\frac{\sin\frac{n\pi t}{2}}{\frac{n\pi}{2}} + \frac{\cos\frac{n\pi t}{2}}{\frac{n^2\pi^2}{4}}\right]_0^1 + 2\left[(2-t)\frac{\sin\frac{n\pi t}{2}}{\frac{n\pi}{2}} - \frac{\cos\frac{n\pi t}{2}}{\frac{n^2\pi^2}{4}}\right]_1^2$$

$$= 2\left[\frac{2}{n\pi}\sin\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{4}{n^2\pi^2} - \frac{4}{n^2\pi^2}\cos n\pi - \frac{2}{n\pi}\sin\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi}{2}\right]$$

$$= 2\left[\frac{8}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{4}{n^2\pi^2} - \frac{4}{n^2\pi^2}\cos n\pi\right]$$

$$a_n = \frac{8}{n^2\pi^2}\left[2\cos\frac{n\pi}{2} - 1 - \cos n\pi\right]$$

For $n = 1$,

$$a_1 = \frac{8}{1^2\pi^2}\left[2\cos\frac{\pi}{2} - 1 - \cos\pi\right] = 0 \qquad \left(\because\ \cos\pi = -1,\ \cos\frac{\pi}{2} = 0\right)$$

For $n = 2$,

$$a_2 = \frac{8}{2^2\pi^2}\left[2\cos\pi - 1 - \cos 2\pi\right] = -\frac{32}{2^2\pi^2} \qquad (\because\ \cos 2\pi = 1)$$

For $n = 3$,

$$a_3 = \frac{8}{3^2\pi^2}\left[2\cos\frac{3\pi}{2} - 1 - \cos 3\pi\right] = 0 \quad (\because \ \cos\frac{3\pi}{2} = 0, \cos 3\pi = -1)$$

For $n = 4$,

$$a_4 = \frac{8}{4^2\pi^2}\left[2\cos 2\pi - 1 - \cos 4\pi\right] = 0$$

For $n = 5$,

$$a_5 = 0$$

For $n = 6$,

$$a_6 = \frac{8}{6^2\pi^2}\left[2\cos 3\pi - 1 - \cos 6\pi\right] = -\frac{32}{6^2\pi^2}$$

$$a_7 = 0,\ a_8 = 0,\ a_9 = 0, a_{10} = -\frac{32}{10^2\pi^2}$$ and so on.

$\therefore$ Half range $\cos ine$ series becomes

$$f(x) = 1 - \frac{32}{\pi^2}\left[\frac{1}{2^2}\cos\pi t + \frac{1}{6^2}\cos 3\pi t + \frac{1}{10^2}\cos 5\pi t + ...\right]$$

Example 15. Obtain the Fourier series for $f(x)$:

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x^2, & 0 < x < \pi \end{cases} \qquad \text{(RGPV Dec 2003)}$$

Sol^n:- The Fourier series be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad .....(1)$$

where $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\,dx + \int_0^{\pi} f(x)\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_0^{\pi} x^2\,dx\right] = \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\ dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0.\cos nx\ dx + \int_0^{\pi} x^2\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left\{x^2\frac{\sin nx}{n}\right\}_0^{\pi} - \int_0^{\pi} 2x\frac{\sin nx}{n}\,dx\right]$$

$$= -\frac{2}{n\pi}\left[-x\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi} = \frac{2}{n^2\pi}\left[\pi\cos n\pi\right] = \frac{2}{n^2}(-1)^n$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0.\sin nx\,dx + \int_{0}^{\pi} x^2 \sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left\{-x^2\frac{\cos nx}{n}\right\}_0^{\pi} + \int_0^{\pi} 2x\frac{\cos nx}{n}dx\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi^2(-1)^n}{n} + \frac{2}{n}\left\{\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right\}_0^{\pi}\right]$$

$$= \frac{(-1)^n}{\pi}\left[\frac{2}{n^3} - \frac{\pi^2}{n}\right]$$

Hence the Fourier series becomes

$$f(x) = \frac{\pi^2}{6} - 2\left[\cos x - \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x - ......\right] -$$

$$\frac{1}{\pi}\left[\left(\frac{2}{1^3} - \frac{\pi^2}{1}\right)\sin x - \left(\frac{2}{2^3} - \frac{\pi^2}{2}\right)\sin 2x + ...\right]$$

Example 16. Obtain the Fourier series for the function:

$$f(x) = \begin{cases} \pi x, & 0 \le x \le 1 \\ \pi(2-x), & 1 \le x \le 2 \end{cases}$$ (RGPV June 2002, Dec 2004)

Sol: The Fourier series be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad .....(1)$$

where

$$a_0 = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx$$

$$= \int_0^1 \pi x\;dx + \int_1^2 \pi(2-x)\,dx$$

$$= \pi\left[\frac{x^2}{2}\right]_0^1 + \pi\left[2x - \frac{x^2}{2}\right]_1^2 = \pi$$

$$a_n = \int_0^1 \pi x \cos nx\, dx + \int_1^2 \pi(2-x)\cos nx\, dx$$

$$= \left[\pi x \frac{\sin n\pi x}{n\pi} - \pi\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right]_0^1 + \left[\pi(2-x)\frac{\sin n\pi x}{n\pi} - (-\pi)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right]_1^2$$

$$= \left(\frac{\cos n\pi}{n^2\pi} - \frac{1}{n^2\pi^2}\right) - \left(\frac{\cos 2n\pi}{n^2\pi} - \frac{\cos n\pi}{n^2\pi}\right)$$

$$= \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

$$b_n = \int_0^1 \pi x \sin n\pi x\, dx + \int_1^2 \pi(2-x)\sin n\pi x\, dx$$

$$= \left[\pi x\left(-\frac{\cos n\pi x}{n\pi}\right) - \pi\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_0^1 + \left[\pi(2-x)\left(-\frac{\cos n\pi x}{n\pi}\right) - (-\pi)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_1^2$$

$$= -\frac{\cos n\pi}{n} + \frac{\cos n\pi}{n} = 0$$

Hence the required Fourier series becomes

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \ldots\ldots\right]$$

Example 17. Obtain a half range $\cos ine$ series for

$$f(x) = \begin{cases} kx, & \text{for } 0 \le x \le \dfrac{l}{2} \\ k(l-x), & \text{for } \dfrac{l}{2} \le x \le l. \end{cases}$$ (RGPV Dec 2004, Feb 2006)

Sol: Let the $\cos ine$ series be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \quad \ldots..(1)$$

where

$$a_0 = \frac{2}{l}\int_0^l f(x)\, dx$$

$$= \frac{2}{l}\left[\int_0^{\frac{l}{2}} kx\, dx + \int_{\frac{l}{2}}^{l} k(l-x)\, dx\right]$$

$$= \frac{2}{l}\left[\left\{\frac{kx^2}{2}\right\}_0^{\frac{l}{2}} + \left\{k\left(lx - \frac{x^2}{2}\right)\right\}_{\frac{l}{2}}^{l}\right]$$

$$= \frac{2}{l}\left[\frac{kl^2}{8} + k\left(l^2 - \frac{l^2}{2}\right) - k\left(\frac{l^2}{2} - \frac{l^2}{8}\right)\right] = \frac{2}{l}\left(\frac{kl^2}{4}\right) = \frac{kl}{2}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\int_0^{\frac{l}{2}} kx.\cos\frac{n\pi x}{l}\,dx + \int_{\frac{l}{2}}^{l} k(l-x)\cos\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{2}{l}\left[\left\{kx\frac{l}{n\pi}\sin\frac{n\pi x}{l} + k\frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right\}_0^{\frac{l}{2}} + \left\{k(l-x)\frac{l}{n\pi}\sin\frac{n\pi x}{l} - k\frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right\}_{\frac{l}{2}}^{l}\right]$$

$$= \frac{2}{l}\left[\left\{\frac{kl^2}{2n\pi}\sin\frac{n\pi}{2} + \frac{kl^2}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right)\right\} + \left\{\frac{-kl^2}{n^2\pi^2}\cos n\pi - \frac{kl^2}{2n\pi}\sin\frac{n\pi}{l} + \frac{kl^2}{n^2\pi^2}\cos\frac{n\pi}{2}\right\}\right]$$

$$= \frac{2}{l}\left[\frac{2kl^2}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{kl^2}{n^2\pi^2} - \frac{kl^2}{n^2\pi^2}\cos n\pi\right]$$

$$a_n = \frac{2kl}{n^2\pi^2}\left[2\cos\frac{n\pi}{2} - 1 - \cos n\pi\right]$$

For $n = 1$,

$$a_1 = \frac{2kl}{1^2\pi^2}\left[2\cos\frac{\pi}{2} - 1 - \cos\pi\right] = 0 \qquad \left(\because \cos\pi = -1, \cos\frac{\pi}{2} = 0\right)$$

For $n = 2$,

$$a_2 = \frac{2kl}{2^2\pi^2}\left[2\cos\pi - 1 - \cos 2\pi\right] = \frac{-8kl}{2^2\pi^2} \qquad (\because \cos 2\pi = 1)$$

For $n = 3$,

$$a_3 = \frac{2kl}{3^2\pi^2}\left[2\cos\frac{3\pi}{2} - 1 - \cos 3\pi\right] = 0 \qquad \left(\because \cos\frac{3\pi}{2} = 0, \cos 3\pi = -1\right)$$

For $n = 4$,

$$a_4 = \frac{2kl}{4^2\pi^2}\left[2\cos 2\pi - 1 - \cos 4\pi\right] = 0$$

For $n=5$,

$$a_5=0,\ a_6=\frac{-8kl}{6^2\pi^2},\ a_7=0,\ a_8=0,\ a_9=0,\ a_{10}=\frac{-8kl}{10^2\pi^2}\text{ and so on.}$$

$\therefore$ Fourier $\cos ine$ series becomes

$$f(x)=\frac{kl}{4}-\frac{8kl}{\pi^2}\left[\frac{1}{2^2}.\cos\frac{2\pi x}{l}+\frac{1}{6^2}.\cos\frac{6\pi x}{l}+\frac{1}{10^2}.\cos\frac{10\pi x}{l}+......\right]$$

Example 18. Expand $f(x)=\sqrt{1-\cos x}$ $,0<x<2\pi$ in a Fourier series. Hence evaluate

$$\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+......$$ (RGPV Sept 2009)

Sol: Let $f(x)=\sqrt{1-\cos x}$ $=\sqrt{2}\sin\frac{x}{2}$

The Fourier series be

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx \qquad .....(1)$$

where

$$a_0=\frac{1}{\pi}\int_0^{2\pi}\sqrt{2}\sin\frac{x}{2}dx$$

$$=\frac{\sqrt{2}}{\pi}\left[-2\cos\frac{x}{2}\right]_0^{2\pi}=\frac{4\sqrt{2}}{\pi}$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}\sqrt{2}\sin\frac{x}{2}\cos nx\ dx$$

$$=\frac{\sqrt{2}}{2\pi}\int_0^{2\pi}2\cos nx\sin\frac{x}{2}dx$$

$$=\frac{1}{\sqrt{2}\pi}\int_0^{2\pi}\left[\sin\left(n+\frac{1}{2}\right)x-\sin\left(n-\frac{1}{2}\right)x\right]dx$$

$$=\frac{1}{\sqrt{2}\pi}\left[\frac{-2}{2n+1}\cos\left(\frac{2n+1}{2}\right)x+\frac{2}{2n-1}\cos\left(\frac{2n-1}{2}\right)x\right]_0^{2\pi}$$

$$= \frac{2}{\sqrt{2}\pi}\left[\frac{-1}{2n+1}\left\{\cos(2n+1)\pi - 1\right\} + \frac{1}{2n-1}\left\{\cos(2n-1)\pi - 1\right\}\right]$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{2}{2n+1} - \frac{2}{2n-1}\right] = \frac{-4\sqrt{2}}{\pi\left(4n^2-1\right)}$$

$$\left(\because \cos(2n+1)\pi = \cos(2n-1)\pi = -1\right)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} \sqrt{2}\sin\frac{x}{2}\sin n\pi x\,dx$$

$$= \frac{\sqrt{2}}{2\pi}\int_0^{2\pi} 2\sin n\pi x\,\sin\frac{x}{2}\,dx$$

$$= \frac{1}{\sqrt{2}\pi}\int_0^{2\pi}\left[\cos\left(n-\frac{1}{2}\right)x - \cos\left(n+\frac{1}{2}\right)x\right]dx$$

$$= \frac{1}{\sqrt{2}\pi}\left[\frac{2}{2n-1}\sin\left(\frac{2n-1}{2}\right)x - \frac{2}{2n+1}\sin\left(\frac{2n+1}{2}\right)x\right]_0^{2\pi}$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{1}{2n-1}\left\{\sin(2n-1)\pi - 0\right\} - \frac{1}{2n+1}\left\{\sin(2n+1)\pi - 0\right\}\right]$$

$$= 0$$

Hence the required Fourier series becomes

$$f(x) = \frac{2\sqrt{2}}{\pi} - \sum_{n=1}^{\infty}\frac{4\sqrt{2}}{\pi\left(4n^2-1\right)}\cos nx$$

when $x = 0$, we have

$$0 = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}$$

$$\text{'or'} \sum_{n=1}^{\infty}\frac{1}{4n^2-1} = \frac{1}{2}$$

$$\text{'or'}\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots\ldots = \frac{1}{2}$$

Example 19. Find the Fourier series expansion of $f(x) = 2x - x^2$ in $(0,3)$ and hence deduce that

$$\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+......=\frac{\pi^2}{12}.$$ (RGPV June 2008)

Sol: The required Fourier series be

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}\text{ where }2l=3 \qquad .....(1)$$

where

$$a_0=\frac{1}{l}\int_0^{2l}f(x)\,dx=\frac{2}{3}\int_0^3\left(2x-x^2\right)dx$$

$$=\frac{2}{3}\left[x^2-\frac{x^3}{3}\right]_0^3=0$$

$$a_n=\frac{1}{l}\int_0^{2l}\left(2x-x^2\right)\cos\frac{n\pi x}{l}\,dx$$

$$=\frac{2}{3}\int_0^3\left(2x-x^2\right)\cos\frac{2n\pi x}{3}\,dx$$

$$=\frac{2}{3}\left[\left(2x-x^2\right)\frac{\cos\frac{2n\pi x}{3}}{\frac{2n\pi}{3}}-(2-2x)\left(\frac{\frac{-\cos 2n\pi x}{3}}{\left(\frac{2n\pi}{3}\right)^2}\right)+(-2)\left(\frac{\frac{-\sin 2n\pi x}{3}}{\left(\frac{2n\pi}{3}\right)^3}\right)\right]_0^3$$

$$=\frac{2}{3}\cdot\frac{9}{4n^2\pi^2}\left[-4\cos 2n\pi-2\right]=\frac{-9}{n^2\pi^2}$$

$$b_n=\frac{1}{l}\int_0^{2l}\left(2x-x^2\right)\sin\frac{n\pi x}{l}\,dx=\frac{2}{3}\int_0^3\left(2x-x^2\right)\sin\frac{2n\pi x}{3}\,dx$$

$$=\frac{2}{3}\left[\left(2x-x^2\right)\left(\frac{\frac{-\cos 2n\pi x}{3}}{\frac{2n\pi}{3}}\right)-(2-2x)\left(\frac{\frac{-\sin 2n\pi x}{3}}{\left(\frac{2n\pi}{3}\right)^2}\right)-2\frac{\frac{\cos 2n\pi x}{3}}{\left(\frac{2n\pi}{3}\right)^3}\right]_0^3$$

$$=\frac{2}{3}\left[\frac{-6}{n^2\pi^2}\cos 2n\pi-\frac{27}{4n^3\pi^3}\left(\cos 2n\pi-1\right)\right]=\frac{3}{n\pi}$$

Hence, the Fourier series becomes

$$2x - x^2 = -\sum_{n=1}^{\infty} \frac{9}{n^2\pi^2} \cos\frac{2n\pi x}{3} + \sum_{n=1}^{\infty} \frac{3}{n\pi} \sin\frac{2n\pi x}{3}$$

Putting $x = \frac{3}{2}$, we have

$$3 - \frac{9}{4} = -\sum_{n=1}^{\infty} \frac{9}{n^2\pi^2} \cos n\pi$$

$$\text{or } \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots = \frac{\pi^2}{12}$$

Example 20. Expand

$$f(x) = \begin{cases} \frac{1}{4} - x, & \text{if } 0 < x < \frac{1}{2} \\ x - \frac{3}{4}, & \text{if } \frac{1}{2} < x < 1 \end{cases}$$

as a Fourier series of $\sin e$ terms. (RGPV Sept 2009)

Sol: The Fourier $\sin e$ series be

$$f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x \qquad \ldots..(1)$$

where

$$b_n = \frac{2}{1}\int_0^1 f(x) \sin n\pi x \, dx$$

$$= 2\left[\int_0^{\frac{1}{2}} \left(\frac{1}{4} - x\right) \sin n\pi x \, dx + \int_{\frac{1}{2}}^1 \left(x - \frac{3}{4}\right) \sin n\pi x \, dx\right]$$

$$= 2\left[-\left(\frac{1}{4} - x\right)\frac{\cos n\pi x}{n\pi} - \frac{\sin n\pi x}{n^2\pi^2}\right]_0^{\frac{1}{2}} + 2\left[-\left(x - \frac{3}{4}\right)\frac{\cos n\pi x}{n\pi} + \frac{\sin n\pi x}{n^2\pi^2}\right]_{\frac{1}{2}}^1$$

$$= 2\left[\frac{1}{4n\pi}\cos\frac{n\pi}{2} + \frac{1}{4n\pi} - \frac{\sin\frac{n\pi}{2}}{n^2\pi^2} - \frac{1}{4n\pi}\cos n\pi - \frac{1}{4n\pi}\cos\frac{n\pi}{2} - \frac{\sin\frac{n\pi}{2}}{n^2\pi^2}\right]$$

$$= \frac{1}{2n\pi}\left[1-(-1)^n\right] - \frac{4\sin\frac{n\pi}{2}}{n^2\pi^2}$$

For $n = 1,\quad b_1 = \frac{1}{\pi} - \frac{4}{\pi^2},$

For $n = 2,\quad b_2 = 0,$

For $n = 3,\quad b_3 = \frac{1}{3\pi} + \frac{4}{3^2\pi^2},$

For $n = 4,\quad b_4 = 0,$

For $n = 5,\quad b_5 = \frac{1}{5\pi} - \frac{4}{5^2\pi^2},$

For $n = 6,\quad b_6 = 0$ and so on.

Hence Fourier sine series becomes

$$f(x) = \left(\frac{1}{\pi} - \frac{4}{\pi^2}\right)\sin \pi x + \left(\frac{1}{3\pi} - \frac{4}{3^2\pi^2}\right)\sin 3\pi x + \left(\frac{1}{5\pi} - \frac{4}{5^2\pi^2}\right)\sin 3\pi x + \ldots\ldots$$

Example 21 If $f(x) = |\cos x|$, expand $f(x)$ as a Fourier series in the interval $(-\pi, \pi)$.

Sol: As $f(-x) = |\cos(-x)| = |\cos x| = f(x)$.

Thus $|\cos x|$ is an even function.

Therefore, the Fourier series be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \qquad \ldots..(1)$$

where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} |\cos x|\, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \cos x\, dx + \int_{\frac{\pi}{2}}^{\pi} (-\cos x)\, dx\right]$$

$$= \frac{2}{\pi}\left[\{\sin x\}_0^{\frac{\pi}{2}} - \{\sin x\}_{\frac{\pi}{2}}^{\pi}\right] = \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} |\cos x| \cos nx\, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \cos x \cos nx\, dx + \int_{\frac{\pi}{2}}^{\pi} (-\cos x)\cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\frac{\pi}{2}} \{\cos(n+1)x + \cos(n-1)x\}\, dx - \int_{\frac{\pi}{2}}^{\pi} \{\cos(n+1)x + \cos(n-1)x\}\, dx\right]$$

$$= \frac{1}{\pi}\left[\left\{\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right\}_0^{\frac{\pi}{2}} - \left\{\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right\}_{\frac{\pi}{2}}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin(n+1)\frac{\pi}{2}}{n+1} + \frac{\sin(n-1)\frac{\pi}{2}}{n-1} + \frac{\sin(n+1)\frac{\pi}{2}}{n+1} + \frac{\sin(n-1)\frac{\pi}{2}}{n-1}\right]$$

$$= \frac{2}{\pi}\left[\frac{\cos\frac{n\pi}{2}}{n+1} - \frac{\cos\frac{n\pi}{2}}{n-1}\right]$$

$$= \frac{-4\cos\frac{n\pi}{2}}{\pi\left(n^2-1\right)}, \quad n \neq 1$$

For $n = 1$, $a_1 = \dfrac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \cos^2 x\, dx - \int_{\frac{\pi}{2}}^{\pi} \cos^2 x\, dx\right] = 0$

Hence Fourier series becomes

$$|\cos x| = \frac{2}{\pi} + \frac{4}{\pi}\left[\frac{1}{3}\cos 2x - \frac{1}{15}\cos 4x + ...\right]$$

Example 22. Prove that in $0 < x < l:\ x = \frac{1}{2} - \frac{4l}{\pi^2}\left[\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + ...\right]$ and deduce that $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + ... = \frac{\pi^4}{96}$. (RGPV Dec 2003)

Sol: Let $f(x) = x,$ then cosine series in $(0, l)$ be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \qquad .....(1)$$

where

$$a_0 = \frac{2}{l}\int_0^l x\,dx = \frac{2}{l}.\frac{l^2}{2} = l$$

$$a_n = \frac{2}{l}\int_0^l x\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\left\{x.\frac{l}{n\pi}\sin\frac{n\pi x}{l}\right\}_0^l - \int_0^l \frac{l}{n\pi}\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{-2}{l}.\frac{l}{n\pi}\left(\frac{-l}{n\pi}\cos\frac{n\pi x}{l}\right)_0^l$$

$$= \frac{2l}{n^2\pi^2}(\cos n\pi - 1) = \frac{2l}{n^2\pi^2}\left[(-1)^n - 1\right]$$

Hence Fourier cosine series becomes

$$x = \frac{l}{2} + \sum_{n=1}^{\infty}\frac{2l}{n^2\pi^2}\left[(-1)^n - 1\right]\cos\frac{n\pi x}{l} \qquad .....(2)$$

$$= \frac{l}{2} - \frac{4l}{\pi^2}\left[\cos\frac{n\pi}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \frac{1}{5^2}\cos\frac{5\pi x}{l} + ...\right]$$

Now, using Parsevel's theorem:

$$\int_a^b \left[f(x) \right]^2 dx = \frac{b-a}{2}\left[\frac{a_0{}^2}{2} + \sum_{n=1}^{\infty} \left(a_n{}^2 + b_n{}^2 \right) \right]$$

$$\Rightarrow \int_0^l x^2 dx = \frac{l-0}{2}\left[\frac{l^2}{2} + \sum_{n=odd}^{\infty} \left(\frac{-4l}{n^2\pi^2} \right)^2 \right]$$

$$\Rightarrow \frac{l^3}{3} = \frac{l}{2}\left[\frac{l^2}{2} - \frac{16l^2}{\pi^4}\left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\ldots \right) \right]$$

$$\Rightarrow \frac{2l^2}{3} = \frac{l^2}{2} - \frac{16l^2}{\pi^4}\left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\ldots \right)$$

$$\Rightarrow \frac{-l^2}{6} = \frac{-16l^2}{\pi^4}\left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\ldots \right)$$

$$\Rightarrow \frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\ldots$$

Example 23. Obtain the Fourier series for the function $f(x) = x^2,\ -\pi \le x \le \pi.$ Hence deduce that

(i) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots\ldots = \frac{\pi^2}{6}$ (RGPV June 2009)

(ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots\ldots = \frac{\pi^2}{12}$

(iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}$

(iv) $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots\ldots = \frac{\pi^4}{90}$ (RGPV June 2008, Dec 2010)

Sol: Let $f(x) = x^2$. *i.e.* an even function then Fourier series be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \qquad \ldots..(1)$$

where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2 dx = \frac{2}{\pi}\left(\frac{x^3}{3} \right)_0^{\pi} = \frac{2}{3}\pi^2$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\, dx$$

$$= \frac{2}{\pi}\left[x^2 \frac{\sin nx}{n} - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[2\pi \frac{\cos n\pi}{n^2}\right] = \frac{4}{n^2}(-1)^n$$

Hence Fourier series becomes

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

$$x^2 = \frac{\pi^2}{3} - 4\left(\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + ...\right) \qquad .....(2)$$

(i) Putting $x = \pi$, we get

$$\pi^2 = \frac{\pi^2}{3} - 4\left(\frac{-1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} - ......\right)$$

$$\Rightarrow \frac{2\pi^2}{3} = 4\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ...\right)$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ... = \frac{\pi^2}{6} \qquad .....(3)$$

(ii) Putting $x = 0$, we get

$$0 = \frac{\pi^2}{3} - 4\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + ...\right)$$

$$\Rightarrow \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + ... = \frac{\pi^2}{12} \qquad .....(4)$$

(iii) Adding (3) and (4), we get

$$2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ...\right) = \frac{\pi^2}{4}$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ... = \frac{\pi^2}{8} \qquad .....(5)$$

(iv) Using Parseval's theorem:

$$\int_{-\pi}^{\pi}\left[f(x)\right]^2 dx = \pi\left[\frac{a_0{}^2}{2}+\sum_{n=1}^{\infty}\left(a_n{}^2+b_n{}^2\right)\right]$$

$$\Rightarrow \int_{-\pi}^{\pi} x^4 dx = \pi\left[\frac{4\pi^4}{18}+\sum_{n=1}^{\infty}\frac{16}{n^4}\right]$$

$$\Rightarrow \left[\frac{x^5}{5}\right]_{-\pi}^{\pi} = \frac{2}{9}\pi^5+\pi\sum_{n=1}^{\infty}\frac{16}{n^4}$$

$$\Rightarrow \frac{2\pi^5}{5}-\frac{2\pi^5}{9} = 16\pi\left(\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+......\right)$$

$$\Rightarrow \frac{8\pi^5}{45} = 16\pi\left(\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+......\right)$$

$$\Rightarrow \frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+...... = \frac{\pi^4}{90}$$

Practice Problems

1. Find the Fourier series representation of $f(x) = x\cos x,\ -\pi < x < \pi.$

$$\left[\text{Ans : } \frac{-1}{2}\sin nx+\sum_{n=2}^{\infty}\frac{2n(-1)^n}{(n^2-1)}\sin nx\right]$$

2. Obtain Fourier series expansion for $f(x) = x\sin x$ in the interval $-\pi < x < \pi$.

   Hence deduce that $\frac{\pi}{4}=\frac{1}{2}+\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}-......$

$$\left[\text{Ans: } 1-\frac{1}{2}\cos x-\frac{2}{1.3}\cos 2x+\frac{2}{2.4}\cos 3x-......\right]$$

3. Find the Fourier series to represent the function $f(x) = |\sin x|,\ -\pi < x < \pi.$

$$\left[\text{Ans: } \frac{2}{\pi}-\frac{4}{\pi}\left[\frac{\cos 2x}{3}+\frac{\cos 4x}{15}+......\right]\right]$$

4. Find the half range cosine series for the function $f(x)=\sin\frac{\pi x}{l},\ 0<x<l.$ (RGPV Dec 2005)

$$\left[\text{Ans: } \frac{2}{\pi}-\frac{4}{\pi}\left[\frac{\cos\frac{2\pi x}{l}}{1.3}+\frac{\cos\frac{4\pi x}{l}}{3.5}+\frac{\cos\frac{6\pi x}{l}}{5.7}+\ldots\ldots\right]\right].$$

5. Obtain the Fourier series for $f(x)=e^{-x}$ in the interval $0<x<2\pi.$

$$\left[\text{Ans: } \frac{1-e^{-2\pi}}{\pi}\left\{\frac{1}{2}+\left(\frac{1}{2}\cos x+\frac{1}{5}\cos 2x+\frac{1}{10}\cos 3x+\ldots\ldots\right)+\left(\frac{1}{2}\sin x+\frac{2}{5}\sin 2x+\frac{3}{10}\sin 3x+\ldots\ldots\right)\right\}\right].$$

6. Find the Fourier series to represent the function

$$f(x)=\begin{cases}0, & -\pi\le x\le 0\\ \sin x, & 0\le x\le \pi\end{cases}$$

Hence show that $\frac{1}{1.3}+\frac{1}{3.5}+\ldots\ldots=\frac{1}{2}$

$$\left[\text{Ans: } \frac{1}{\pi}-\frac{2}{\pi}\left[\frac{1}{3}\cos 2x+\frac{1}{15}\cos 4x+\ldots\ldots\right]+\frac{1}{2}\sin x\right].$$

7. Find the Fourier series expansion of the periodic function of period 2π, defined by

$$f(x)=\begin{cases}x, & \text{if } \frac{-\pi}{2}<x<\frac{\pi}{2}\\ \pi-x, & \text{if } \frac{\pi}{2}<x<\frac{3\pi}{2}.\end{cases}$$

$$\left[\text{Ans: } \frac{4}{\pi}\left[\frac{\sin x}{1^2}-\frac{\sin 3x}{3^2}+\frac{\sin 5x}{5^2}-\ldots\ldots\right]\right].$$

8. Expand the function $f(x)$ in Fourier series in the interval $(-\pi,\pi)$:

$$f(x)=\begin{cases}2x, & 0\le x\le \pi\\ x, & -\pi\le x\le 0.\end{cases}$$

$$\left[\text{Ans: } \frac{\pi}{4}-\frac{2}{\pi}\left[\frac{1}{1^2}\cos x+\frac{1}{3^2}\cos 3x+\ldots\ldots\right]+3\left[\sin x-\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x-\ldots\ldots\right]\right]$$

9. If $f(x)$ is a function defined by

$$f(x)=\begin{cases} x, & 0\le x\le \dfrac{\pi}{2} \\ \pi-x, & \dfrac{\pi}{2}\le x\le \pi \end{cases}$$

Express $f(x)$ by a sine Fourier series and also by a cosine series.

$$\left[\text{Ans: } \frac{4}{\pi}\left[\frac{\sin x}{1^2}-\frac{\sin 3x}{3^2}+\frac{\sin 5x}{5^2}-\ldots\ldots\right], \frac{\pi}{4}-\frac{8}{\pi}\left[\frac{\cos 2x}{2^2}+\frac{\cos 6x}{6^2}+\frac{\cos 10x}{10^2}+\ldots\ldots\right]\right].$$

10. Find the Fourier expansion of the periodic function

$$f(x)=\begin{cases} -k, & -\pi<x<0 \\ k, & 0<x<\pi \end{cases}$$

and $f(x+2\pi)=f(x)$. Sketch the graph of $f(x)$.

$$\left[\text{Ans: } \frac{4k}{\pi}\left[\frac{\sin x}{1}+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+\ldots\ldots\right]\right].$$

11. Expand $f(x)$ in a Fourier series in the interval $(0,2)$ if

$$f(x)=\begin{cases} x, & 0<x<1 \\ 0, & 1<x<2 \end{cases}$$

$$\left[\text{Ans: } \frac{1}{4}-\frac{2}{\pi^2}\left[\cos \pi x+\frac{1}{3^2}\cos 3\pi x+\frac{1}{5^2}\cos 5\pi x+\ldots\ldots\right]+\frac{1}{\pi}\left[\sin \pi x-\frac{\sin 2\pi x}{2}+\frac{\sin 3\pi x}{3}-\ldots\ldots\right]\right]$$

12. Find the Fourier cosine series for $f(x)=x(\pi-x)$ in $0<x<\pi$ and use Parseval's theorem to prove that

$$\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90}$$

13. Find the Fourier series expansion for the function $f(x) = x - x^2$ in $-1 < x < 1$.

$$\left[\text{Ans: } \frac{-1}{3} + \frac{4}{\pi^2}\left[\frac{\cos \pi x}{1^2} - \frac{\cos 2\pi x}{2^2} + \frac{\cos 3\pi x}{3^2} - ......\right] + \frac{2}{\pi}\left[\frac{\sin x}{1^2} - \frac{\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3} - ......\right]\right]$$

14. Find the Fourier series for the function

$$f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ 1, & 0 < x < \dfrac{\pi}{2} \\ 0, & \dfrac{\pi}{2} \le x \le \pi \end{cases}$$

$$\left[\text{Ans: } \frac{1}{4} + \frac{2}{\pi}\left[\frac{\cos x}{1} - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} - ......\right] + \frac{1}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + ...\right]\right].$$

15. Express cosh$x$ in Fourier series in the interval $-\pi < x < \pi$.

$$\left[\text{Ans: } \frac{2}{\pi}\sinh \pi\left[\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{(-1)^n - 1}{n^2 + 1}\right)\cos nx\right]\right].$$

16. Obtain Fourier series for the function $f(x)$ given by

$$f(x) = \begin{cases} 1 + \dfrac{2x}{\pi}, & -\pi \le x \le 0 \\ 1 - \dfrac{2x}{\pi}, & 0 \le x \le \pi \end{cases}$$

Hence, deduce that $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + ... = \dfrac{\pi^2}{8}$.